Variational time stepping method for analytical system dynamics models

Claude Lacoursière*, Tomas Sjöström†,

HPC2N/UMIT
University of Umeå
SE-901 87, Umeå, Sweden
* claude@hpc2n.umu.se
† <tomas_sjostrom@hotmail.com>

Abstract

We introduce a variational time-stepping scheme to integrate the Differential Algebraic Equations (DAE)s of finite-dimensional multi-domain systems. To do this we use the analytical system dynamics framework [1] which is based directly on a constrained Lagrangian formulation, by contrast to models based purely on differential equations. With this formulation, we can naturally couple multibody system dynamics to other components such as electronics, hydraulics, and drivelines among other, using natural boundary conditions, kinematic constraints, and conservation laws for couplings the different components. The time discretization is then performed using the methods of discrete-time variational mechanics [2].

Analytical system dynamics is an application of Lagrangian mechanics for finite dimensional systems meaning that the equations of motion are derived from the least action principle. Each component in a system appears in the Lagrange function as a kinetic store, a potential store, or a dissipation potential. For example, a point mass m with velocity v is kinetic store with energy $\frac{1}{2}mv^2$, a spring with strength k and extension x is a potential store $\frac{1}{2}kx^2$, and a damping force with magnitude $f=-\gamma v$ is a dissipation store $\frac{1}{2}\gamma v^2$. Likewise for electrical circuits, we have inductance, capacitance, and resistance corresponding to the mechanical elements, and for fluids, we have pipes, containers, and fittings. Mechanical systems are subject to kinematic constraints, electrical systems are subject to Kirchhoff's law, and fluid power systems are subject to mass conservation, and this corresponds to holonomic and non-holonomic constraints. Using q and $\dot{q}$ as generalized positions and velocities, and using $T(q,\dot{q}),U(q)$, and $\mathfrak{R}(q,\dot{q})$ respectively for the agglomerated kinetic stores, potential stores, and dissipation potentials, we have $\mathscr{L}(q,\dot{q})=T(q,\dot{q})-U(q)$. Introducing holonomic constraints $g(q)$ with Jacobian $G=\partial g/\partial q$, as well as non-holonomic constraints $A(q,\dot{q})\dot{q}=0$, d'Alembert's principle reads motion

$$\delta\int_0^T \mathrm{d}s\mathscr{L}(q,\dot{q})-\int_0^T \mathrm{d}s\delta q(s)\frac{\partial\mathfrak{R}(q,\dot{q})}{\partial\dot{q}}=0, \tag{1}$$

where δq is an infinitesimal variation of the trajectory $(q(t),\dot{q}(t)),t\in[0,T]$. The DAEs of motion for this are well known

$$\frac{\partial^2 T(q,\dot{q})}{\partial\dot{q}\partial\dot{q}^T}\ddot{q}+\frac{\partial\mathfrak{R}(q,\dot{q})}{\partial\dot{q}}+\frac{\partial U(q)}{\partial q}-G^T\lambda-A^T\nu=\frac{\partial^2 T(q,\dot{q})}{\partial q\partial\dot{q}^T}\dot{q},\ \text{with}\ ,\ g(q)=0,\ \text{and}\ A(q,\dot{q})\dot{q}=0, \tag{2}$$

where λ and ν are Lagrange multipliers. These equations are well behaved for mechanical systems as long as the mass matrix $M=\frac{\partial^2 T}{\partial\dot{q}\partial\dot{q}^T}$ has full rank. However, in electrical circuits for instance, this matrix can have very low rank as neither resistors or capacitors have mass. This means that the Hessians of the energy functions $\frac{\partial^2 T(q,\dot{q})}{\partial\dot{q}\partial\dot{q}^T}$, $\frac{\partial^2 U(q)}{\partial q\partial q^T}$ and $\frac{\partial^2\mathfrak{R}}{\partial\dot{q}\partial\dot{q}^T}$ can, and generally are, degenerate. In consequence, the numerical time integration of Eqn. (2) can be difficult, and involves expensive numerical methods such as singular value decomposition, for instance [3]. This is hardly usable for real-time simulation, and not even scalable by any measure.

Discrete-time variational integrators [2] provide an interesting solution. The principle is to discretize Eqn. (1) directly. To do this, the following is used.

$$S[q]=\int_0^T \mathrm{d}s\mathscr{L}(q,\dot{q})=\sum_{k=0}^{N}\int_{kh}^{(k+1)h}\mathrm{d}q\mathscr{L}(q,\dot{q})=\sum_{k=0}^{N}\mathbb{L}_d(q_k,q_{k+1}),\ \text{and}$$

$$\int_0^h \mathrm{d}s f\cdot\delta q(s)=f_d^{(+)}(q_0,q_1)\delta q_0+f_d^{(-)}(q_0,q_1)\delta q_1 \tag{3}$$

$$D_1\mathbb{L}_d(q_k,q_{k+1})+D_2\mathbb{L}_d(q_{k-1},q_k)+f_d^{(+)}(q_k,q_{k+1})+f_d^{(-)}(q_{k-1},q_k)+G^T\lambda+A^T\nu=0$$

$$g(q_{k+1})=0,\ \text{and}\ A(q_{k+1},q_k)(q_{k+1}-q_k)/h=0.$$

The main idea here is to choose different quadratures in the first and second line of Eqn. (3). When using choices such as $\dot{q} = (q_{k+1} - q_k)/h$, and $\int_0^h \mathrm{d}s U(q) \approx hU(\frac{q_1+q_0}{2})$ and $f_d^{(-)}(q_1, q_0) = 0$, and $f_d^{(+)}(q_0, q_1) = -\frac{\partial \mathfrak{R}(q_{k+1},(q_{k+1}-q_k)/h)}{\partial \dot{q}}$, we obtain implicit integrators and in such cases, we obtain "numerical mass" for all elements. Kinetic stores already have mass, but potential stores and dissipation potentials now have pseudo masses $\tilde{M}_p = \frac{h^2}{4}\frac{\partial^2 U(q)}{\partial q \partial q^T}$ and $\tilde{M}_d = h\frac{\partial^2 \mathfrak{R}}{\partial \dot{q} \partial \dot{q}^T}$. Collecting all masses and pseudo masses together, the quasilinear stepping equations become, writing $v_{k+1} = (q_{k+1} - q_k)/h$,

$$\begin{bmatrix} \tilde{M} & -G_k^T & -A_k^T \\ G_{k'} & 0 & 0 \\ A_{k'} & 0 & 0 \end{bmatrix} \begin{bmatrix} v_{k+1} \\ \lambda \\ \nu \end{bmatrix} = \begin{bmatrix} Mv_k + hf \\ \zeta \\ 0 \end{bmatrix}, \text{ and } \quad q_{k+1} = q_k + hv_{k+1}. \tag{4}$$

We agglomerated all the "soft" forces in f_e, i.e., forces for which pseudo-mass is not necessary. The "numerical" mass matrix $\tilde{M}$ is then non-degenerate and one can use standard sparse matrix packages to solve the linear problems. Note also that gyroscopic forces require special treatment and usually involve added mass of sort [4]. The index k' indicates that we could perform Newton Raphson iteration to reach $g_{k+1} = 0$, and $A(q_{k+1}, v_{k+1})v_{k+1} = 0$. The term ζ is there to contain constraint stabilization. In our own simulation, we use $\zeta = -(4\tau/h)g_k + \tau G_k v_k$, and we also add perturbations on the diagonal of the matrix for stability. The parameter τ is further defined as $1/(1+4d/h)$, where $d \approx 2$ is a damping rate. This parameterization was presented previously and is linearly stable [5]. We do not perform Newton-Raphson iterations either as we found that our constraint stabilization is reliable and stable enough to yield good results, comparable to other simulation tools which use variable time-step and higher order methods.

Non-smooth dynamics can then be included in this time-stepping scheme [6].

Using this technique, we have simulated electronics systems with diodes and transistors, and these were coupled via electric motors to multibody system and to hydraulic systems. We have modeled pistons as three-body kinematic constraints, making the fluid volume proportional to the separation distance between two rigid bodies coupled with a prismatic joint. We have also implemented drivelines which use simple rotational elements instead of full rigid bodies.

A full validation of our results is presented elsewhere [7]. We use this method for real-time simulation of heavy ground vehicle as shown below in Fig. 1.

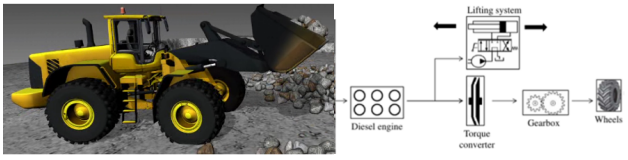

Figure 1: A full vehicle simulation

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