

COMPUTATIONAL METHODS FOR PLASTICITY

# COMPUTATIONAL METHODS FOR PLASTICITY

## THEORY AND APPLICATIONS

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