A simple example

Let $\mathscr{X}$ now be a space of sufficiently smooth[¶] functions $x : \mathscr{R}^n \to \mathscr{R}$. We define a *functional* $y : \mathscr{X} \to \mathscr{R}$ as

$$y(x) \equiv \int_\Omega \sin(x(\boldsymbol{p}))\, \mathrm{d}\boldsymbol{p}, \tag{2.153}$$

where $\Omega \subset \mathscr{R}^n$ is a given integration domain. Linearisation of the above functional about a given argument (function) x_0 is the following generalisation of (2.114):

$$l(u) = y(x_0) + \mathrm{D}y(x_0)\,[u] = \int_\Omega \sin(x_0(\boldsymbol{p}))\, \mathrm{d}\boldsymbol{p} + \mathrm{D}y(x_0)\,[u], \tag{2.154}$$

where the directional derivative $\mathrm{D}y(x_0)\,[u]$ is now a linear transformation on the *function* $u \in \mathscr{X}$ and can be determined by direct generalisation of (2.113):

$$\begin{aligned} \mathrm{D}y(x_0)\,[u] &= \frac{\mathrm{d}}{\mathrm{d}\epsilon} y(x_0 + \epsilon u)\bigg|_{\epsilon=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}\epsilon} \int_\Omega \sin(x_0(\boldsymbol{p}) + \epsilon\, u(\boldsymbol{p}))\, \mathrm{d}\boldsymbol{p}\bigg|_{\epsilon=0} \\ &= \int_\Omega \cos(x_0(\boldsymbol{p}))\, u(\boldsymbol{p})\, \mathrm{d}\boldsymbol{p}. \end{aligned} \tag{2.155}$$

From the above, the linearisation of the functional (2.153) at x_0 is then established as

$$l(u) = \int_\Omega \sin(x_0(\boldsymbol{p}))\, \mathrm{d}\boldsymbol{p} + \int_\Omega \cos(x_0(\boldsymbol{p}))\, u(\boldsymbol{p})\, \mathrm{d}\boldsymbol{p}. \tag{2.156}$$

[¶]To avoid a precise statement of regularity properties of functions, we frequently use the term *sufficiently smooth* in the present text, meaning that functions have a sufficient degree of regularity so that all operations in which they are involved are properly defined.

3 ELEMENTS OF CONTINUUM MECHANICS AND THERMODYNAMICS

THIS chapter reviews some basic concepts of mechanics and thermodynamics of continuous media. The definitions and notation introduced will be systematically employed throughout the subsequent chapters of this book. The material presented here is well established in the continuum mechanics literature and an effort has been made to follow the notation and nomenclature in use in standard textbooks (Billington and Tate, 1981; Bonet and Wood, 1997; Ciarlet, 1988; Gurtin, 1981; Lemaitre and Chaboche, 1990; Ogden, 1984; Spencer, 1980; Truesdell and Noll, 1965).

3.1. Kinematics of deformation

Let $\mathscr{B}$ be a *body* which occupies an open region Ω of the three-dimensional Euclidean space $\mathscr{E}$ with a regular boundary $\partial\Omega$ in its *reference* configuration. A *deformation* of $\mathscr{B}$ (Figure 3.1) is defined by a smooth one-to-one function

$$\boldsymbol{\varphi} : \Omega \to \mathscr{E}$$

that maps each material particle[†] $\boldsymbol{p}$ of $\mathscr{B}$ into a point

$$\boldsymbol{x} = \boldsymbol{\varphi}(\boldsymbol{p}) \tag{3.1}$$

where the particle is positioned in the *deformed* configuration of $\mathscr{B}$. The region of $\mathscr{E}$ occupied by $\mathscr{B}$ in its deformed configuration will be denoted

$$\boldsymbol{\varphi}(\Omega).$$

The vector field $\boldsymbol{u}(\boldsymbol{p})$, defined by

$$\boldsymbol{u}(\boldsymbol{p}) = \boldsymbol{\varphi}(\boldsymbol{p}) - \boldsymbol{p}, \tag{3.2}$$

is the *displacement* of $\boldsymbol{p}$. Thus, one may write

$$\boldsymbol{x} = \boldsymbol{p} + \boldsymbol{u}(\boldsymbol{p}). \tag{3.3}$$

[†]For convenience, material particles of $\mathscr{B}$ will be identified with their positions in the reference configuration of $\mathscr{B}$.

Computational Methods for Plasticity: Theory and Applications EA de Souza Neto, D Perić and DRJ Owen

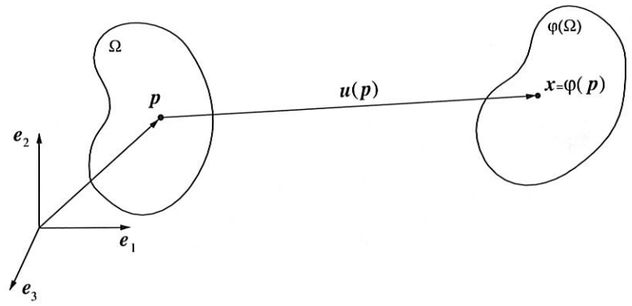

Figure 3.1. Deformation.

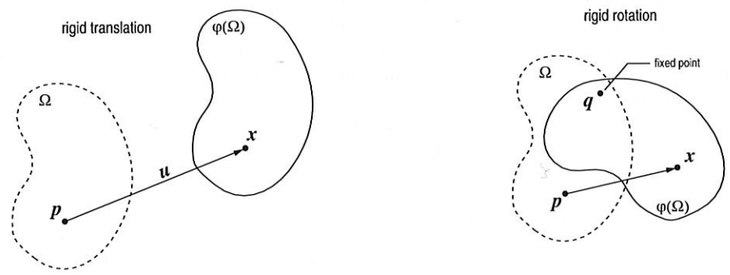

Figure 3.2. Rigid deformations.

A *rigid deformation* of $\mathscr{B}$ is a deformation that preserves the distances between all material particles of $\mathscr{B}$. A rigid deformation (Figure 3.2) can be a translation, a rotation, or a combination of a translation and a rotation. A *rigid translation* is a deformation with constant displacement vector ($\boldsymbol{u}$ independent of $\boldsymbol{p}$):

$$\boldsymbol{\varphi}(\boldsymbol{p}) = \boldsymbol{p} + \boldsymbol{u}. \tag{3.4}$$

A *rigid rotation* is a deformation that can be expressed as

$$\boldsymbol{\varphi}(\boldsymbol{p}) = \boldsymbol{q} + \boldsymbol{R}\,(\boldsymbol{p} - \boldsymbol{q}), \tag{3.5}$$

where $\boldsymbol{R}$ is a proper orthogonal tensor (a rotation) and $\boldsymbol{q}$ is the point about which $\mathscr{B}$ is rotated. A deformation is rigid, including translations and/or rotations, if and only if it can be expressed in the form:

$$\boldsymbol{\varphi}(\boldsymbol{p}) = \boldsymbol{\varphi}(\boldsymbol{q}) + \boldsymbol{R}\,(\boldsymbol{p} - \boldsymbol{q}). \tag{3.6}$$

The deformation map above represents a rigid translation with displacement $\boldsymbol{\varphi}(\boldsymbol{q}) - \boldsymbol{q}$ superimposed on a rigid rotation $\boldsymbol{R}$ about point $\boldsymbol{q}$.

A time-dependent deformation of $\mathscr{B}$ is called a *motion* of $\mathscr{B}$. A motion (Figure 3.3) is defined by a function

$$\boldsymbol{\varphi} : \Omega \times \mathscr{R} \to \mathscr{E},$$

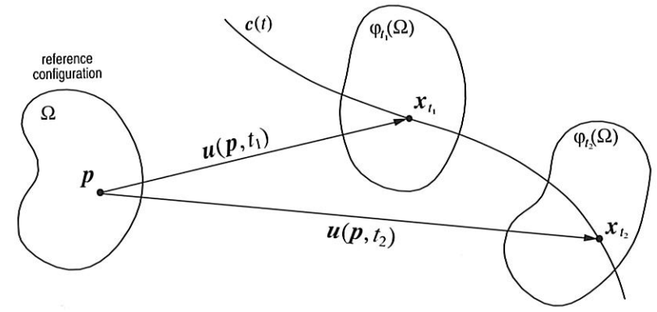

Figure 3.3. Motion.

so that for each time t, the map $\boldsymbol{\varphi}(\cdot, t)$ is a deformation of $\mathscr{B}$. The deformation map at time t will be also denoted $\boldsymbol{\varphi}_t$. During the motion $\boldsymbol{\varphi}$, the position $\boldsymbol{x}$ of a material particle $\boldsymbol{p}$ at time t is given by

$$\boldsymbol{x} = \boldsymbol{\varphi}(\boldsymbol{p}, t). \tag{3.7}$$

Similarly,

$$\boldsymbol{\varphi}(\Omega, t)$$

will denote the region of $\mathscr{E}$ occupied by the body $\mathscr{B}$ at time t. In terms of the displacement field the motion is expressed as

$$\boldsymbol{\varphi}(\boldsymbol{p}, t) = \boldsymbol{p} + \boldsymbol{u}(\boldsymbol{p}, t). \tag{3.8}$$

The parametric curve $\boldsymbol{c}(t)$, defined as

$$\boldsymbol{c}(t) = \boldsymbol{\varphi}(\boldsymbol{p}, t) \tag{3.9}$$

for a *fixed* material point $\boldsymbol{p}$, describes the *trajectory* of $\boldsymbol{p}$ during the motion of $\mathscr{B}$.

During a motion $\boldsymbol{\varphi}$, the *velocity* of a material particle $\boldsymbol{p}$ is defined by

$$\dot{\boldsymbol{x}}(\boldsymbol{p}, t) = \frac{\partial \boldsymbol{\varphi}(\boldsymbol{p}, t)}{\partial t}. \tag{3.10}$$

Since at each time t the map $\boldsymbol{\varphi}(\cdot, t)$ is one-to-one (and hence invertible) by assumption, material points can be expressed in terms of the place they occupy at a time t as

$$\boldsymbol{p} = \boldsymbol{\varphi}^{-1}(\boldsymbol{x}, t) = \boldsymbol{x} - \boldsymbol{u}(\boldsymbol{\varphi}^{-1}(\boldsymbol{x}, t), t). \tag{3.11}$$

The map $\boldsymbol{\varphi}^{-1}$ is called the *reference map*. Using the reference map, one may define the function

$$\boldsymbol{v}(\boldsymbol{x}, t) \equiv \dot{\boldsymbol{x}}(\boldsymbol{\varphi}^{-1}(\boldsymbol{x}, t), t). \tag{3.12}$$

The field $\boldsymbol{v}$ is called the *spatial velocity* and gives the velocity of the material particle positioned at $\boldsymbol{x}$ at time t.

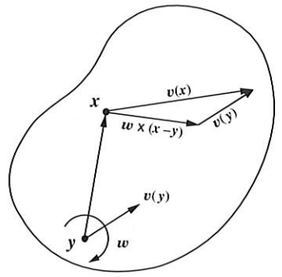

Figure 3.4. Rigid velocity.

A *rigid motion* of $\mathscr{B}$ is a motion for which, at each time t, the map $\varphi(\cdot, t)$ is a rigid deformation. A motion φ is rigid if and only if at each time t, the spatial velocity v admits the representation

$$v(x, t) = v(y, t) + W(t)\,(x - y) \tag{3.13}$$

for all $x, y \in \varphi(\Omega, t)$, with $W(t)$ a *skew* tensor. The velocity at x is given as the sum of a uniform velocity $v(y, t)$ and a superimposed rotation about the line that passes through y and is parallel to the axial vector associated to the skew tensor W. By denoting $w(t)$ the axial vector of $W(t)$, the velocity field above can be re-written as

$$v(x, t) = v(y, t) + w(t) \times (x - y), \tag{3.14}$$

which is the standard formula for the velocity field of classical rigid-body dynamics. The vector $w(t)$ is called the *angular velocity* of the body. The rigid velocity field is schematically illustrated in Figure 3.4.

3.1.1. MATERIAL AND SPATIAL FIELDS

Both fields $\dot{x}$ and v introduced above describe the velocity of material particles. However, $\dot{x}$ and v have different arguments. While $\dot{x}$ has material particle and time as arguments, the arguments of v are spatial position and time. This motivates the following definitions: Let a general time-dependent (scalar, vectorial or tensorial) field α be defined over the body $\mathscr{B}$. If the domain of α is $\Omega \times \mathscr{R}$, i.e. if the value of α is expressed as a function of material particles p (and time) then α is said to be a *material field*. On the other hand, if its domain is $\varphi_t(\Omega) \times \mathscr{R}$, then α is said to be a *spatial field*. Using (3.7), the *material description* of a spatial field $\alpha(x, t)$ is defined by

$$\alpha_m(p, t) = \alpha(\varphi(p, t), t). \tag{3.15}$$

Conversely, the *spatial description* of a material field $\beta(p, t)$ is defined by

$$\beta_s(x, t) = \beta(\varphi^{-1}(x, t), t). \tag{3.16}$$

It should be noted that *any* field associated with a motion of $\mathscr{B}$ can be expressed as a function of time and material particles *or* spatial position. A material (spatial) field does

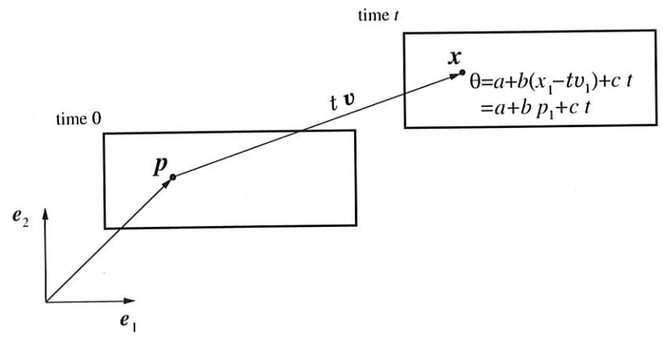

Figure 3.5. Material and spatial descriptions.

not necessarily represent a quantity physically associated with the reference (deformed) configuration of the body.

Example 3.1.1. Consider, for instance, the rectangular body of Figure 3.5 subjected to the *rigid translation*:

$$x = \varphi(p, t) \equiv p + t\,v,$$

with constant velocity v. Assume that, during the motion φ, the temperature field of the body in question is linearly distributed along its longitudinal axis and varies uniformly throughout the body at a constant rate. Taking the initial configuration (at $t = 0$) as the reference configuration (and, therefore, labelling material particles of the body with their position p at time 0), the material description of this temperature field reads

$$\theta_m(p, t) = a + b\,p_1 + c\,t,$$

where a, b and c are constants. In view of the assumed motion φ, the spatial description of the same field is given by

$$\theta_s(x, t) = \theta_m(p(x, t), t) = a + b(x_1 - t\,v_1) + c\,t.$$

Note that, in spite of having p as one of its arguments, θ_m (as θ_s) expresses a physical quantity associated with the configuration of time t. The spatial description θ_s gives the temperature, at time t, of the material particle whose position at time t is x. In experimental terms, it would be the temperature read from a thermometer held fixed in space at x. The function θ_m gives the temperature, at time t, of the material particle whose position at time 0 is p. It would be the temperature indicated by a thermometer attached to this material particle.

To avoid notational complexity, the subscripts m and s employed above to denote the material and spatial descriptions of general fields will not be used throughout this book unless absolutely necessary. In general, the description employed will be evident either from the context or from the argument used (p or x).

3.1.2. MATERIAL AND SPATIAL GRADIENTS, DIVERGENCES AND TIME DERIVATIVES

The *material* and *spatial gradients* of a general field α, denoted respectively $\nabla_p \alpha$ and $\nabla_x \alpha$, are defined as

$$\nabla_p \alpha = \frac{\partial}{\partial \boldsymbol{p}} \alpha_m(\boldsymbol{p}, t), \quad \nabla_x \alpha = \frac{\partial}{\partial \boldsymbol{x}} \alpha_s(\boldsymbol{x}, t), \tag{3.17}$$

i.e. they are, respectively, the derivatives of α with respect to $\boldsymbol{p}$ and $\boldsymbol{x}$ holding t fixed.

Similarly, the *material* and *spatial time derivatives* of α, denoted respectively $\dot{\alpha}$ and α', are defined by

$$\dot{\alpha} = \frac{\partial}{\partial t} \alpha_m(\boldsymbol{p}, t), \quad \alpha' = \frac{\partial}{\partial t} \alpha_s(\boldsymbol{x}, t). \tag{3.18}$$

The material time derivative $\dot{\alpha}$ measures the rate of change of α at a *fixed material particle* $\boldsymbol{p}$. The spatial time derivative, on the other hand, measures the rate of change of α observed at a *fixed spatial position* $\boldsymbol{x}$. In the example of Figure 3.5, the material and spatial time derivatives of the temperature field θ are given by

$$\dot{\theta} = c, \quad \theta' = -b\, v_1 + c.$$

The material time derivative in this case corresponds to the temperature rate computed from a thermometer attached to a material particle $\boldsymbol{p}$ whilst θ' is the temperature rate observed in a thermometer held fixed in space at $\boldsymbol{x}$. Note that the extra term $-b\, v_1$ added to θ' is a contribution to the rate of change of temperature at $\boldsymbol{x}$ due to the motion of the body combined with its non-uniform distribution of temperature. This contribution vanishes if the body moves parallel to $\boldsymbol{e}_2$ ($v_1 = 0$), i.e. the direction of temperature isolines. It would also vanish if the temperature were uniform throughout the body ($b = 0$).

Analogously to (2.145) (page 37), we define the *spatial* and *material divergence* of a vector field $\boldsymbol{v}$, respectively, as

$$\mathrm{div}_p\, \boldsymbol{v} = \mathrm{tr}(\nabla_p \boldsymbol{v}), \quad \mathrm{div}_x\, \boldsymbol{v} = \mathrm{tr}(\nabla_x \boldsymbol{v}). \tag{3.19}$$

In addition (refer to (2.147)), for a tensor field $\boldsymbol{T}$, the spatial and material divergence are given, in Cartesian components, by

$$(\mathrm{div}_x\, \boldsymbol{T})_i = \frac{\partial T_{ij}}{\partial x_j}, \quad (\mathrm{div}_p\, \boldsymbol{T})_i = \frac{\partial T_{ij}}{\partial p_j}. \tag{3.20}$$

The compact definition (2.146) is also applicable to the material and spatial divergence of a tensor.

3.1.3. THE DEFORMATION GRADIENT

The *deformation gradient* of the motion $\boldsymbol{\varphi}$ is the second-order tensor $\boldsymbol{F}$ defined by

$$\boldsymbol{F}\,(\boldsymbol{p}, t) = \nabla_p \boldsymbol{\varphi}(\boldsymbol{p}, t) = \frac{\partial \boldsymbol{x}_t}{\partial \boldsymbol{p}}. \tag{3.21}$$

In view of (3.8) it can be written as

$$\boldsymbol{F} = \boldsymbol{I} + \nabla_p \boldsymbol{u}. \tag{3.22}$$

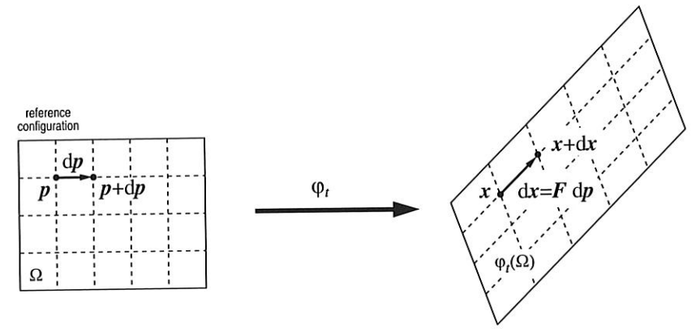

Figure 3.6. The deformation gradient.

The Cartesian components of $\boldsymbol{F}$ are given by

$$\begin{aligned} F_{ij} &= \frac{\partial x_i}{\partial p_j} \\ &= \delta_{ij} + \frac{\partial u_i}{\partial p_j}, \end{aligned} \tag{3.23}$$

where x_i denote the components of $\boldsymbol{x}_t$. In terms of the reference map (3.11), the deformation gradient may be equivalently expressed as

$$\boldsymbol{F}\,(\boldsymbol{x}, t) = [\nabla_x \boldsymbol{\varphi}^{-1}(\boldsymbol{x}, t)]^{-1} = [\boldsymbol{I} - \nabla_x \boldsymbol{u}]^{-1}. \tag{3.24}$$

Consider the infinitesimal material fibre $\mathrm{d}\boldsymbol{p}$ that connects two neighbouring material particles $\boldsymbol{p}$ and $\boldsymbol{p} + \mathrm{d}\boldsymbol{p}$ of a deforming body (Figure 3.6). Under the deformation $\boldsymbol{\varphi}_t$, these particles are mapped, respectively, into $\boldsymbol{x}$ and $\boldsymbol{x} + \mathrm{d}\boldsymbol{x}$. The deformation gradient is the linear operator that relates infinitesimal material fibres $\mathrm{d}\boldsymbol{p}$ with their deformed counterparts $\mathrm{d}\boldsymbol{x}$:

$$\mathrm{d}\boldsymbol{x} = \boldsymbol{F}\ \mathrm{d}\boldsymbol{p}. \tag{3.25}$$

A deformation of $\mathscr{B}$ with uniform deformation gradient ($\boldsymbol{F}$ independent of $\boldsymbol{p}$) is called a *homogeneous deformation*. A deformation is homogeneous if and only if it admits the representation

$$\boldsymbol{\varphi}(\boldsymbol{p}) = \boldsymbol{\varphi}(\boldsymbol{q}) + \boldsymbol{F}\,(\boldsymbol{p} - \boldsymbol{q}) \tag{3.26}$$

for all points $\boldsymbol{p}$, $\boldsymbol{q} \in \mathscr{B}$, with $\boldsymbol{F}$ a positive definite tensor. Clearly, rigid translations and rotations are homogeneous deformations.

3.1.4. VOLUME CHANGES. THE DETERMINANT OF THE DEFORMATION GRADIENT

Consider now the infinitesimal volume $\mathrm{d}v_0$ defined by the infinitesimal vectors $\mathrm{d}\boldsymbol{a}$, $\mathrm{d}\boldsymbol{b}$ and $\mathrm{d}\boldsymbol{c}$ emanating from the material particle $\boldsymbol{p}$ in the reference configuration (Figure 3.7). Trivially,

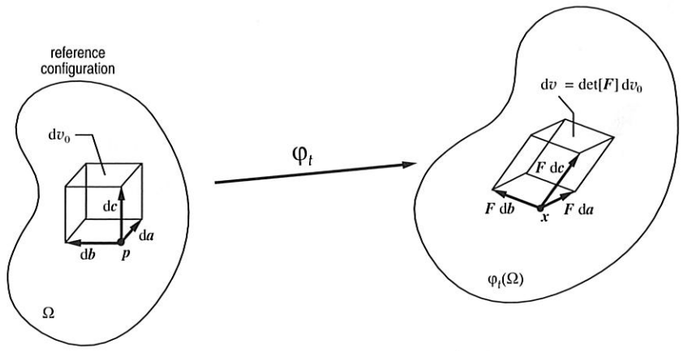

Figure 3.7. The determinant of the deformation gradient.

one has

$$\mathrm{d}v_0 = (\mathrm{d}\boldsymbol{a} \times \mathrm{d}\boldsymbol{b}) \cdot \mathrm{d}\boldsymbol{c}. \tag{3.27}$$

The deformation φ_t maps the infinitesimal vectors, respectively, into $\boldsymbol{F}\,\mathrm{d}\boldsymbol{a}$, $\boldsymbol{F}\,\mathrm{d}\boldsymbol{b}$ and $\boldsymbol{F}\,\mathrm{d}\boldsymbol{c}$, so that the deformed infinitesimal volume is given by

$$\mathrm{d}v = (\boldsymbol{F}\,\mathrm{d}\boldsymbol{a} \times \boldsymbol{F}\,\mathrm{d}\boldsymbol{b}) \cdot \boldsymbol{F}\,\mathrm{d}\boldsymbol{c}. \tag{3.28}$$

By making use of identity (2.54), it follows that

$$\det \boldsymbol{F} = \frac{\mathrm{d}v}{\mathrm{d}v_0}, \tag{3.29}$$

i.e. the determinant of the deformation gradient represents, locally, the volume after deformation per unit reference volume (or *volume change ratio*). Throughout this book, we will adopt the following notation

$$J \equiv \det \boldsymbol{F}. \tag{3.30}$$

From (3.29) it follows that if $\det \boldsymbol{F} = 0$, then the infinitesimal volume has collapsed into a material particle. Since the body is not allowed to penetrate itself (this restriction is embodied in the assumption that the deformation map is one-to-one), this represents a physically unacceptable situation. Also note that, at the reference configuration, $\boldsymbol{F} = \boldsymbol{I}$ and, consequently, $J = 1$. Thus, a configuration with $J < 0$ cannot be reached from the reference configuration without having, at some stage, $J = 0$. Therefore, in any deformed configuration of a body, J satisfies

$$J > 0. \tag{3.31}$$

Isochoric deformations

Isochoric (or *volume-preserving*) deformations are deformations that do not produce changes in volume. A locally isochoric deformation is characterised by

$$J = 1. \tag{3.32}$$

Volumetric deformations

Volumetric deformations (i.e. *pure contractions/dilations*) are deformations consisting purely of a uniform contraction/dilation in all directions. The deformation gradient of any volumetric deformation is a spherical tensor:

$$\boldsymbol{F} = \alpha\, \boldsymbol{I}, \tag{3.33}$$

where the *scalar* α is the corresponding contraction/dilation ratio. With l_0 and l denoting, respectively, the undeformed and deformed lengths of a material fibre, for a locally volumetric deformation we have:

$$\frac{l}{l_0} = \alpha \tag{3.34}$$

in *all directions*.

3.1.5. ISOCHORIC/VOLUMETRIC SPLIT OF THE DEFORMATION GRADIENT

Any deformation can be locally decomposed as a purely volumetric deformation followed by an isochoric deformation or as an isochoric deformation followed by a pure volumetric deformation. To see this, note that the deformation gradient can always be multiplicatively split as

$$\boldsymbol{F} = \boldsymbol{F}_{\mathrm{iso}}\, \boldsymbol{F}_{\mathrm{v}} = \boldsymbol{F}_{\mathrm{v}}\, \boldsymbol{F}_{\mathrm{iso}}, \tag{3.35}$$

where

$$\boldsymbol{F}_{\mathrm{v}} \equiv (\det \boldsymbol{F})^{\frac{1}{3}}\, \boldsymbol{I} \tag{3.36}$$

is the *volumetric* component of $\boldsymbol{F}$ and

$$\boldsymbol{F}_{\mathrm{iso}} \equiv (\det \boldsymbol{F})^{-\frac{1}{3}}\, \boldsymbol{F} \tag{3.37}$$

is the *isochoric* (*volume-preserving* or *unimodular*) component. Note that, by construction, $\boldsymbol{F}_{\mathrm{v}}$ corresponds indeed to a purely volumetric deformation (it has the representation (3.33)) and, since

$$\det \boldsymbol{F}_{\mathrm{v}} = [(\det \boldsymbol{F})^{\frac{1}{3}}]^3 \det \boldsymbol{I} = \det \boldsymbol{F}, \tag{3.38}$$

$\boldsymbol{F}_{\mathrm{v}}$ produces the same volume change as $\boldsymbol{F}$. The isochoric component in turn represents a volume preserving deformation, that is,

$$\det \boldsymbol{F}_{\mathrm{iso}} = [(\det \boldsymbol{F})^{-\frac{1}{3}}]^3 \det \boldsymbol{F} = 1. \tag{3.39}$$

3.1.6. POLAR DECOMPOSITION. STRETCHES AND ROTATION

By applying the *polar decomposition* to the deformation gradient, one obtains:

$$\boldsymbol{F} = \boldsymbol{R}\boldsymbol{U} = \boldsymbol{V}\boldsymbol{R}, \tag{3.40}$$

where the proper orthogonal tensor $\boldsymbol{R}$ is the local *rotation tensor* and the symmetric positive definite tensors $\boldsymbol{U}$ and $\boldsymbol{V}$ are, respectively, the *right* and *left stretch tensors*. The right and left stretch tensors are related by the rotation

$$\boldsymbol{V} = \boldsymbol{R}\boldsymbol{U}\boldsymbol{R}^T. \tag{3.41}$$

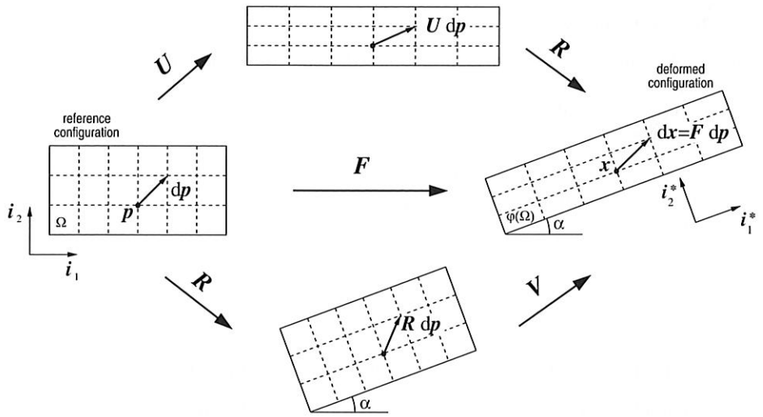

Figure 3.8. Polar decomposition of the deformation gradient. Stretches and rotation.

The stretch tensors $\boldsymbol{U}$ and $\boldsymbol{V}$ can be expressed as

$$\boldsymbol{U} = \sqrt{\boldsymbol{C}}, \quad \boldsymbol{V} = \sqrt{\boldsymbol{B}}, \tag{3.42}$$

where $\boldsymbol{C}$ and $\boldsymbol{B}$ – named, respectively, the *right* and *left Cauchy–Green strain tensors* – are defined by

$$\boldsymbol{C} = \boldsymbol{U}^2 = \boldsymbol{F}^T\boldsymbol{F}, \quad \boldsymbol{B} = \boldsymbol{V}^2 = \boldsymbol{F}\,\boldsymbol{F}^T. \tag{3.43}$$

Example 3.1.2 (A simple plane deformation). To illustrate the meaning of the polar decomposition of $\boldsymbol{F}$, a simple example consisting of a body subjected to a homogeneous deformation, i.e. with $\boldsymbol{F}$ independent of $\boldsymbol{p}$, is given in what follows. Consider the rectangular body of Figure 3.8 subjected to homogeneous stretching/compression in the directions of its longitudinal and transversal axes (respectively, the directions of $\boldsymbol{i}_1$ and $\boldsymbol{i}_2$ in the reference configuration) with a superimposed rigid rotation of angle α. With p_i and x_i denoting coordinates of $\boldsymbol{p}$ and $\boldsymbol{x}$ in the Cartesian system associated with the orthonormal basis $\{\boldsymbol{i}_1, \boldsymbol{i}_2\}$, the deformation map is defined as

$$\varphi : \begin{cases} x_1 = p_1\,\lambda_1\,\cos\alpha - p_2\,\lambda_2\,\sin\alpha \\ x_2 = p_1\,\lambda_1\,\sin\alpha + p_2\,\lambda_2\,\cos\alpha, \end{cases} \tag{3.44}$$

where the factors λ_1 and λ_2 determine how much stretching/compression occurs, respectively, along the longitudinal and transversal axes. In the basis $\{\boldsymbol{i}_1, \boldsymbol{i}_2\}$, the matrix representation of the corresponding deformation gradient is given by

$$\boldsymbol{F} = \begin{bmatrix} \lambda_1\,\cos\alpha & -\lambda_2\,\sin\alpha \\ \lambda_1\,\sin\alpha & \lambda_2\,\cos\alpha \end{bmatrix}. \tag{3.45}$$

The rotation tensor, obtained from the polar decomposition of $\boldsymbol{F}$, is represented by

$$\boldsymbol{R} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \tag{3.46}$$

and the right and left stretch tensors by

$$\boldsymbol{U} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \tag{3.47}$$

and

$$\boldsymbol{V} = \begin{bmatrix} \lambda_1\,\cos^2\alpha + \lambda_2\,\sin^2\alpha & (\lambda_1 - \lambda_2)\,\sin\alpha\,\cos\alpha \\ (\lambda_1 - \lambda_2)\,\sin\alpha\,\cos\alpha & \lambda_1\,\sin^2\alpha + \lambda_2\,\cos^2\alpha \end{bmatrix}. \tag{3.48}$$

Insight into the meaning of the polar decomposition of the deformation gradient can be gained by focusing now on the generic infinitesimal fibre represented by $\mathrm{d}\boldsymbol{p}$ in Figure 3.8. Under deformation, $\mathrm{d}\boldsymbol{p}$ is mapped into $\mathrm{d}\boldsymbol{x} = \boldsymbol{F}\,\mathrm{d}\boldsymbol{p}$. With use of the polar decomposition of $\boldsymbol{F}$, this mapping can be split into two sequential steps. If the *right* polar decomposition $\boldsymbol{F} = \boldsymbol{R}\,\boldsymbol{U}$ is used, the two steps are:

1. $\mathrm{d}\boldsymbol{p} \longrightarrow \boldsymbol{U}\,\mathrm{d}\boldsymbol{p}$,
2. $\boldsymbol{U}\,\mathrm{d}\boldsymbol{p} \longrightarrow \boldsymbol{R}\,(\boldsymbol{U}\,\mathrm{d}\boldsymbol{p}) = \boldsymbol{F}\,\mathrm{d}\boldsymbol{p}$.

In the first operation, $\mathrm{d}\boldsymbol{p}$ deforms as if the body were being purely stretched (or compressed) along the directions of its longitudinal and transversal axes (which at this stage coincide with $\boldsymbol{i}_1$ and $\boldsymbol{i}_2$ respectively). The second mapping is a pure rotation (of angle α) of the deformed fibre $\boldsymbol{U}\,\mathrm{d}\boldsymbol{p}$ and corresponds to a rigid rotation of the body. If the *left* polar decomposition $\boldsymbol{F} = \boldsymbol{V}\boldsymbol{R}$ is employed instead, the sequence is reversed:

1. $\mathrm{d}\boldsymbol{p} \longrightarrow \boldsymbol{R}\,\mathrm{d}\boldsymbol{p}$,
2. $\boldsymbol{R}\,\mathrm{d}\boldsymbol{p} \longrightarrow \boldsymbol{V}(\boldsymbol{R}\,\mathrm{d}\boldsymbol{p}) = \boldsymbol{F}\,\mathrm{d}\boldsymbol{p}$.

In this case, the fibre is first rigidly rotated by an angle α. The second operation corresponds to the deformation of the fibre under pure stretching/compression of the body along its axial and transversal directions. However, due to the previous rotation, these directions coincide now with $\boldsymbol{i}_1^* = \boldsymbol{R}\,\boldsymbol{i}_1$ and $\boldsymbol{i}_2^* = \boldsymbol{R}\,\boldsymbol{i}_2$, respectively. Note that if the basis $\{\boldsymbol{i}_1^*, \boldsymbol{i}_2^*\}$ is used, the matrix representation of $\boldsymbol{V}$ reads

$$\boldsymbol{V} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \tag{3.49}$$

so that the transformation $(\cdot) \to \boldsymbol{V}(\cdot)$ indeed corresponds to stretchings along the directions of $\boldsymbol{i}_1^*$ and $\boldsymbol{i}_2^*$.

The above example has illustrated the significance of the polar decomposition of $\boldsymbol{F}$. The discussion has been restricted to a homogeneous deformation only to ease visualisation of the stretches and rotation involved in the decomposition of the deformation gradient. It should be

remarked that for a generic deformation of a body, in which $\boldsymbol{F}$ is a function of $\boldsymbol{p}$, intermediate configurations of the body corresponding to pure stretching or pure rigid rotation (such as those illustrated in Figure 3.8) do not exist in general. Nevertheless, the interpretation of $\boldsymbol{U}$ and $\boldsymbol{V}$ as pure stretchings and of $\boldsymbol{R}$ as a rigid rotation remain valid in a *local* sense. Note that for *any* deformation φ, one may write:

$$\boldsymbol{x} + \mathrm{d}\boldsymbol{x} = \varphi(\boldsymbol{p} + \mathrm{d}\boldsymbol{p}) = \boldsymbol{p} + \boldsymbol{F}(\boldsymbol{p})\,\mathrm{d}\boldsymbol{p}, \tag{3.50}$$

that is, within an infinitesimal neighbourhood of a material point $\boldsymbol{p}$, the deformation behaves like a homogeneous deformation with gradient $\boldsymbol{F}(\boldsymbol{p})$. Thus, within this infinitesimal neighbourhood of $\boldsymbol{p}$, $\boldsymbol{U}(\boldsymbol{p})$ and $\boldsymbol{V}(\boldsymbol{p})$ measure stretches from $\boldsymbol{p}$ and $\boldsymbol{R}(\boldsymbol{p})$ measures the local rigid rotation.

Spectral decomposition of the stretch tensors

Since $\boldsymbol{U}$ and $\boldsymbol{V}$ are symmetric, it follows from the *spectral theorem* that they admit the spectral decomposition

$$\boldsymbol{U} = \sum_{i=1}^{3} \lambda_i\, \boldsymbol{l}_i \otimes \boldsymbol{l}_i, \quad \boldsymbol{V} = \sum_{i=1}^{3} \lambda_i\, \boldsymbol{e}_i \otimes \boldsymbol{e}_i, \tag{3.51}$$

where the $\{\lambda_1, \lambda_2, \lambda_3\}$ are the eigenvalues of $\boldsymbol{U}$ (and $\boldsymbol{V}$) named the *principal stretches*. The vectors $\boldsymbol{l}_i$ and $\boldsymbol{e}_i$ are unit eigenvectors of $\boldsymbol{U}$ and $\boldsymbol{V}$ respectively. The triads $\{\boldsymbol{l}_1, \boldsymbol{l}_2, \boldsymbol{l}_3\}$ and $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ form orthonormal bases for the space $\mathscr{V}$ of vectors in $\mathscr{E}$. They are called, respectively, the *Lagrangian* and *Eulerian triads* and define the *Lagrangian* and *Eulerian principal directions*.

Substitution of (3.41) into (3.51) gives the following relationship between the eigenvectors of $\boldsymbol{V}$ and $\boldsymbol{U}$:

$$\boldsymbol{l}_i = \boldsymbol{R}\,\boldsymbol{e}_i, \tag{3.52}$$

that is, each vector $\boldsymbol{e}_i$ differs from the corresponding $\boldsymbol{l}_i$ by a rotation $\boldsymbol{R}$.

The spectral decomposition of the right and left stretch tensors implies that in *any* deformation, the local stretching from a material particle can always be expressed as a superposition of stretches along three mutually orthogonal directions. In the example discussed above, illustrated by Figure 3.8, $\{\lambda_1, \lambda_2\}$ are the principal stretches and the Lagrangian and Eulerian bases are, respectively, $\{\boldsymbol{i}_1, \boldsymbol{i}_2\}$ and $\{\boldsymbol{i}_1^*, \boldsymbol{i}_2^*\}$.

3.1.7. STRAIN MEASURES

In the above section, we have seen that in a local sense, i.e. within an infinitesimal neighbourhood of a generic material particle $\boldsymbol{p}$, pure rotations can be distinguished from pure stretching by means of the polar decomposition of the deformation gradient. Under the action of pure rotations, the distances between particles within this neighbourhood remain fixed. When the distances between material particles are identical to their values in the reference configuration, we say that the region surrounding $\boldsymbol{p}$ is *unstrained*. In this case, the difference between the deformed neighbourhood of $\boldsymbol{p}$ and its reference configuration is a rigid deformation. Pure stretching, on the other hand, characterised by $\boldsymbol{U}$ or $\boldsymbol{V}$, changes the

distance between material particles. Under stretching, we say that the region surrounding $\boldsymbol{p}$ is *strained*. To quantify straining, i.e. to evaluate how much $\boldsymbol{U}$ (or $\boldsymbol{V}$) departs from $\boldsymbol{I}$ (a rigid deformation), some kind of *strain measure* needs to be defined.

Let us consider, again, the generic material fibre represented by the infinitesimal vector $\mathrm{d}\boldsymbol{p}$ that emanates from $\boldsymbol{p}$ (Figure 3.8 serves as an illustration). The deformation maps $\mathrm{d}\boldsymbol{p}$ into $\mathrm{d}\boldsymbol{x} = \boldsymbol{F}\,\mathrm{d}\boldsymbol{p}$. Thus, the square of the deformed length of the material fibre in question reads

$$\|\mathrm{d}\boldsymbol{x}\|^2 = \boldsymbol{F}\,\mathrm{d}\boldsymbol{p} \cdot \boldsymbol{F}\,\mathrm{d}\boldsymbol{p} = \boldsymbol{C}\,\mathrm{d}\boldsymbol{p} \cdot \mathrm{d}\boldsymbol{p} = (\boldsymbol{I} + 2\,\boldsymbol{E}^{(2)})\,\mathrm{d}\boldsymbol{p} \cdot \mathrm{d}\boldsymbol{p}, \tag{3.53}$$

where $\boldsymbol{C} = \boldsymbol{F}^T\,\boldsymbol{F} = \boldsymbol{U}^2$ is the right Cauchy–Green tensor and the *strain measure* $\boldsymbol{E}^{(2)}$ (the meaning of the superscript will be made clear below) is the so-called *Green–Lagrange* strain tensor *defined* as

$$\begin{aligned} \boldsymbol{E}^{(2)} &= \tfrac{1}{2}(\boldsymbol{C} - \boldsymbol{I}) \\ &= \tfrac{1}{2}[\nabla_p \boldsymbol{u} + (\nabla_p \boldsymbol{u})^T + (\nabla_p \boldsymbol{u})^T \nabla_p \boldsymbol{u}]. \end{aligned} \tag{3.54}$$

No straining occurs, that is, the size of any infinitesimal material fibre emanating from $\boldsymbol{p}$ remains constant ($\|\mathrm{d}\boldsymbol{x}\| = \|\mathrm{d}\boldsymbol{p}\|, \ \forall\ \mathrm{d}\boldsymbol{p}$), if and only if $\boldsymbol{E}^{(2)} = \boldsymbol{0}$. This condition is equivalent to $\boldsymbol{C} = \boldsymbol{U} = \boldsymbol{I}$, implying that $\boldsymbol{F}$ is an orthogonal tensor and the deformation is rigid (pure translation and/or rotation) in the neighbourhood of $\boldsymbol{p}$. From the definition of $\boldsymbol{E}^{(2)}$, its eigenvectors coincide with the Lagrangian triad so that it can be expressed as

$$\boldsymbol{E}^{(2)} = \sum_{i=1}^{3} \tfrac{1}{2}(\lambda_i^2 - 1)\,\boldsymbol{l}_i \otimes \boldsymbol{l}_i, \tag{3.55}$$

and, since it measures strains along the principal *Lagrangian* directions, it is called a *Lagrangian strain measure*.

It must be emphasised that the Green–Lagrange strain measure is *defined* by expression (3.54). It is by no means the unique way of quantifying straining. In fact, the definition of a strain measure is somewhat arbitrary and a specific choice is usually dictated by mathematical and physical convenience. An important family of *Lagrangian strain tensors*, i.e. strain measures based on the Lagrangian triad, is defined by Seth (1964), Hill (1978) and Ogden (1984)

$$\boldsymbol{E}^{(m)} = \begin{cases} \dfrac{1}{m}(\boldsymbol{U}^m - \boldsymbol{I}) & m \neq 0 \\ \ln[\boldsymbol{U}] & m = 0 \end{cases} \tag{3.56}$$

where m is a real number and $\ln[\,\cdot\,]$ denotes the *tensor logarithm* of $[\,\cdot\,]$. Equivalently, in terms of its spectral decomposition, (3.56) may be rephrased as

$$\boldsymbol{E}^{(m)} = \sum_{i=1}^{3} f(\lambda_i)\,\boldsymbol{l}_i \otimes \boldsymbol{l}_i, \tag{3.57}$$

where

$$f(\lambda_i) = \begin{cases} \dfrac{1}{m}(\lambda_i^m - 1) & m \neq 0 \\ \ln \lambda_i & m = 0. \end{cases} \tag{3.58}$$

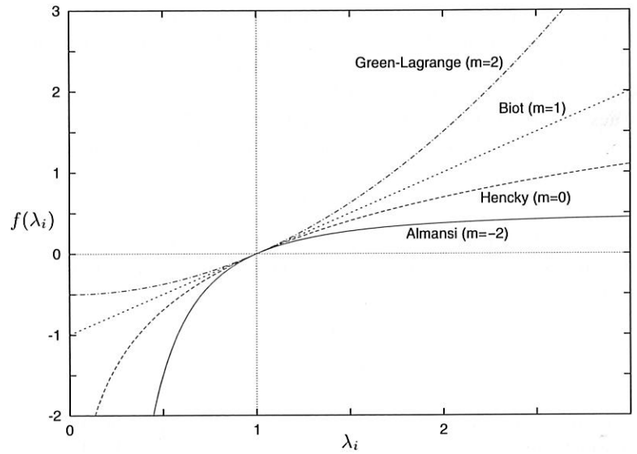

Figure 3.9. Strain measures. Principal strain as a function of the principal stretch for various strain measures.

The Green–Lagrange strain tensor, $\boldsymbol{E}^{(2)}$, is a particular member of this family (with $m = 2$). Other commonly used members of this family are the Biot ($m = 1$), Hencky ($m = 0$) and Almansi ($m = -2$) strain tensors. Note that for any m, the associated strain tensor vanishes if and only if the deformation gradient represents, locally, a rigid deformation, i.e.

$$\boldsymbol{E}^{(m)} = \boldsymbol{0} \iff \boldsymbol{U} = \boldsymbol{I} \iff \boldsymbol{F} = \boldsymbol{R}. \tag{3.59}$$

To illustrate the relationship between the stretch and strain tensors, the principal strain for various strain measures is plotted in Figure 3.9 as a function of the corresponding principal stretch.

Analogously to the strain measures discussed above, it is also possible to define tensors that measure strain along the principal *Eulerian* directions or, simply, *Eulerian strain tensors*. Based on the *left* stretch tensor, the Eulerian counterpart of the Lagrangian family of strain measures above is defined by

$$\boldsymbol{\varepsilon}^{(m)} = \begin{cases} \dfrac{1}{m}(\boldsymbol{V}^m - \boldsymbol{I}) & m \neq 0 \\ \ln[\boldsymbol{V}] & m = 0, \end{cases} \tag{3.60}$$

or, using the Eulerian triad,

$$\boldsymbol{\varepsilon}^{(m)} = \sum_{i=1}^{3} f(\lambda_i)\, \boldsymbol{e}_i \otimes \boldsymbol{e}_i. \tag{3.61}$$

Lagrangian and Eulerian strain tensors are related by

$$\boldsymbol{\varepsilon}^{(m)} = \boldsymbol{R}\, \boldsymbol{E}^{(m)} \boldsymbol{R}^T, \tag{3.62}$$

that is, they differ by the local rotation $\boldsymbol{R}$.

3.1.8. THE VELOCITY GRADIENT. RATE OF DEFORMATION AND SPIN

The spatial field $\boldsymbol{L}$, defined as

$$\boldsymbol{L} = \nabla_x \boldsymbol{v}, \tag{3.63}$$

is named the *velocity gradient*. Equivalently, with application of the chain rule one has

$$\boldsymbol{L} = \frac{\partial}{\partial t}\left(\frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{p}}\right)\frac{\partial \boldsymbol{p}}{\partial \boldsymbol{x}} = \dot{\boldsymbol{F}}\, \boldsymbol{F}^{-1}. \tag{3.64}$$

Two important tensors are obtained by splitting $\boldsymbol{L}$ into its symmetric and skew parts. Namely, the *rate of deformation* tensor (also referred to as the *stretching* tensor), $\boldsymbol{D}$, and the *spin* tensor, $\boldsymbol{W}$, are defined by

$$\boldsymbol{D} = \mathrm{sym}(\boldsymbol{L}), \quad \boldsymbol{W} = \mathrm{skew}(\boldsymbol{L}). \tag{3.65}$$

To gain insight into the physical meaning of the tensors $\boldsymbol{D}$ and $\boldsymbol{W}$, it is convenient to consider a body undergoing a motion with uniform (independent of $\boldsymbol{x}$) velocity gradient. For such a motion the velocity field reads

$$\boldsymbol{v}(\boldsymbol{x}, t) = \boldsymbol{v}(\boldsymbol{y}, t) + \boldsymbol{L}(t)\,(\boldsymbol{x} - \boldsymbol{y}). \tag{3.66}$$

If the decomposition of $\boldsymbol{L}$ into its symmetric and skew parts is introduced, the velocity field can be split as

$$\boldsymbol{v}(\boldsymbol{x}, t) = \boldsymbol{v}^R(\boldsymbol{x}, t) + \boldsymbol{v}^S(\boldsymbol{x}, t), \tag{3.67}$$

where the following definitions have been used:

$$\begin{aligned} \boldsymbol{v}^R(\boldsymbol{x}, t) &= \boldsymbol{v}(\boldsymbol{y}, t) + \boldsymbol{W}(t)\,(\boldsymbol{x} - \boldsymbol{y}), \\ \boldsymbol{v}^S(\boldsymbol{x}, t) &= \boldsymbol{D}(t)\,(\boldsymbol{x} - \boldsymbol{y}). \end{aligned} \tag{3.68}$$

By recalling expression (3.13), the velocity $\boldsymbol{v}^R$, associated with the spin tensor $\boldsymbol{W}$, can be immediately identified as a *rigid* velocity. The only contribution to straining is then provided by the term $\boldsymbol{v}^S$, associated with the rate of deformation tensor. Note that, due to its symmetry, $\boldsymbol{D}$ admits the representation

$$\boldsymbol{D} = \sum_{i=1}^{3} d_i\, \boldsymbol{e}_i \otimes \boldsymbol{e}_i, \tag{3.69}$$

with d_i and $\{\boldsymbol{e}_i\}$, respectively, the eigenvalues and an orthonormal basis of eigenvectors of $\boldsymbol{D}$. With the spectral representation above, the velocity field $\boldsymbol{v}^S$ can be decomposed as a sum of three linearly independent velocities of the form:

$$d_i\,(\boldsymbol{e}_i \otimes \boldsymbol{e}_i)\,(\boldsymbol{x} - \boldsymbol{y}),$$

with no summation implied on i, so that the components of $\boldsymbol{v}^S$ relative to the basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ are given by

$$v_i^S = d_i\,(x_i - y_i), \tag{3.70}$$

again with no summation implied, where x_i and y_i denote the coordinates of points $\boldsymbol{x}$ and $\boldsymbol{y}$ in a Cartesian system associated to $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$. As schematically illustrated in Figure 3.10,

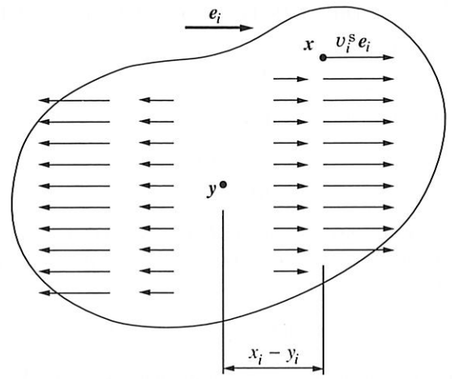

Figure 3.10. Straining velocity field.

each v_i^S corresponds to a velocity field that purely stretches the body in the direction of $\boldsymbol{e}_i$, with the plane perpendicular to $\boldsymbol{e}_i$ that passes through $\boldsymbol{y}$ fixed. Thus, the rate of deformation tensor corresponds indeed to a pure stretching of the body.

If a general motion (in which $\boldsymbol{L}$ is not necessarily uniform) is considered, the above decomposition of the velocity field into the sum of a rigid velocity and a straining velocity remains valid in the *local* sense. In this case, consider a point $\boldsymbol{x}$ and a point $\boldsymbol{x} + \mathrm{d}\boldsymbol{x}$ lying within an infinitesimal neighbourhood of $\boldsymbol{x}$. The velocity field within this infinitesimal neighbourhood of $\boldsymbol{x}$ is given by

$$\boldsymbol{v}(\boldsymbol{x} + \mathrm{d}\boldsymbol{x}, t) = \boldsymbol{v}(\boldsymbol{x}, t) + \boldsymbol{L}(\boldsymbol{x}, t)\, \mathrm{d}\boldsymbol{x}, \tag{3.71}$$

so that, in any motion, the velocity field can be *locally* decomposed as a sum of a rigid velocity

$$\boldsymbol{v}(\boldsymbol{x}, t) + \boldsymbol{W}(\boldsymbol{x}, t)\, \mathrm{d}\boldsymbol{x},$$

associated with the spin tensor $\boldsymbol{W}$, and a straining velocity

$$\boldsymbol{D}(\boldsymbol{x}, t)\, \mathrm{d}\boldsymbol{x},$$

associated exclusively to the rate of deformation tensor $\boldsymbol{D}$.

3.1.9. RATE OF VOLUME CHANGE

The *rate of volume change*, $\dot{J}$, is related to the rate of deformation tensor through the expression

$$\dot{J} = J \operatorname{tr} \boldsymbol{D}. \tag{3.72}$$

To derive this expression, we first apply the chain rule to obtain

$$\dot{J} \equiv (\det \boldsymbol{F})^{\cdot} = \frac{\partial(\det \boldsymbol{F})}{\partial \boldsymbol{F}} : \dot{\boldsymbol{F}} = J\, \boldsymbol{F}^{-T} : \dot{\boldsymbol{F}}, \tag{3.73}$$

where we have made use of relation (2.140) (page 36) for the derivative of the determinant. This, together with definition (2.36) (page 22) of the trace of a tensor and the fact that the skew symmetry of $\boldsymbol{W}$ implies

$$\operatorname{tr} \boldsymbol{L} = \operatorname{tr} \boldsymbol{D}, \tag{3.74}$$

leading to (3.72).

Also note that from the definition (2.145) of the divergence of a vector field we have

$$\operatorname{tr} \boldsymbol{D} = \operatorname{div}_x \boldsymbol{v}, \tag{3.75}$$

so that the rate of volume change can be equivalently expressed as

$$\dot{J} = J \operatorname{div}_x \boldsymbol{v}. \tag{3.76}$$

3.2. Infinitesimal deformations

Small or infinitesimal deformations are deformations with sufficiently small displacement gradient, $\nabla_p \boldsymbol{u}$. For such deformations, the description of kinematics can be substantially simplified.

3.2.1. THE INFINITESIMAL STRAIN TENSOR

Recall definition (3.43) of the Cauchy–Green tensors. In terms of the displacement gradient, one has

$$\begin{aligned} \boldsymbol{C} &= \boldsymbol{I} + \nabla_p \boldsymbol{u} + (\nabla_p \boldsymbol{u})^T + (\nabla_p \boldsymbol{u})^T\, \nabla_p \boldsymbol{u}, \\ \boldsymbol{B} &= \boldsymbol{I} + \nabla_p \boldsymbol{u} + (\nabla_p \boldsymbol{u})^T + \nabla_p \boldsymbol{u}\, (\nabla_p \boldsymbol{u})^T. \end{aligned} \tag{3.77}$$

If the displacement gradient is sufficiently small, the second-order terms in $\nabla_p \boldsymbol{u}$ of the expressions above can be neglected so that, under small deformations, the following approximation can be made

$$\boldsymbol{C} \approx \boldsymbol{B} \approx \boldsymbol{I} + \nabla_p \boldsymbol{u} + (\nabla_p \boldsymbol{u})^T. \tag{3.78}$$

From the above expression and the definitions of the Green–Lagrange strain tensor $\boldsymbol{E}^{(2)}$ and its Eulerian counterpart $\boldsymbol{\varepsilon}^{(2)}$, it follows that, to the same order of approximation,

$$\boldsymbol{E}^{(2)} \approx \boldsymbol{\varepsilon}^{(2)} \approx \tfrac{1}{2}[\nabla_p \boldsymbol{u} + (\nabla_p \boldsymbol{u})^T]. \tag{3.79}$$

This motivates the definition of the *infinitesimal strain tensor* to measure strains under small deformations

$$\boldsymbol{\varepsilon} \equiv \nabla_p^s \boldsymbol{u}, \tag{3.80}$$

where we have introduced the notation

$$\nabla^s(\cdot) = \operatorname{sym}[\nabla(\cdot)] = \tfrac{1}{2}[\nabla(\cdot) + \nabla(\cdot)^T], \tag{3.81}$$

for the *symmetric gradient* of a vector field. It is worth pointing out here that $\boldsymbol{\varepsilon}$ is a *linear* functional of $\boldsymbol{u}$. This fact greatly simplifies the description of small deformations.

In fact, it can be easily shown that not only $\boldsymbol{E}^{(2)}$ and $\boldsymbol{\varepsilon}^{(2)}$ but *all* Lagrangian and Eulerian strain measures defined by expressions (3.56) and (3.60) have the same small deformation limit, i.e. for any m and to within an error of second order in $\nabla_p \boldsymbol{u}$, one has

$$\boldsymbol{\varepsilon}^{(m)} \approx \boldsymbol{E}^{(m)} \approx \boldsymbol{\varepsilon}. \tag{3.82}$$

3.2.2. INFINITESIMAL RIGID DEFORMATIONS

In terms of the infinitesimal strain tensor, the square of the deformed length of a generic material fibre $\mathrm{d}\boldsymbol{p}$ (recall the text preceding expression (3.53)) reads

$$\|\mathrm{d}\boldsymbol{x}\|^2 = (\boldsymbol{I} + 2\,\boldsymbol{\varepsilon})\,\mathrm{d}\boldsymbol{p}\cdot\mathrm{d}\boldsymbol{p} + o(\nabla_p \boldsymbol{u}) \tag{3.83}$$

with $o(\nabla_p \boldsymbol{u})$ a term of second order in $\nabla_p \boldsymbol{u}$. It is clear from this expression that, to within an error of $o(\nabla_p \boldsymbol{u})$, only the symmetric part $\boldsymbol{\varepsilon}$ of $\nabla_p \boldsymbol{u}$ is associated with local straining. The skew part of $\nabla_p \boldsymbol{u}$ produces no straining and is associated exclusively with local infinitesimal rigid rotations. For a pure local infinitesimal rigid rotation ($\|\mathrm{d}\boldsymbol{x}\| = \|\mathrm{d}\boldsymbol{p}\|, \ \forall\ \|\mathrm{d}\boldsymbol{p}\|$) the tensor $\boldsymbol{\varepsilon}$ vanishes or, equivalently, $\nabla_p \boldsymbol{u}$ is skew.

For a body under an arbitrary homogeneous deformation ($\nabla_p \boldsymbol{u}$ independent of $\boldsymbol{p}$), the displacement field can be written as

$$\boldsymbol{u}(\boldsymbol{p}) = \boldsymbol{u}(\boldsymbol{q}) + \nabla_p \boldsymbol{u}\,(\boldsymbol{p} - \boldsymbol{q}), \tag{3.84}$$

for all points $\boldsymbol{p}$ and $\boldsymbol{q}$. For infinitesimal rigid deformations and within an approximation of second order in the displacement gradient, $\nabla_p \boldsymbol{u}$ is skew and the field $\boldsymbol{u}$ can be written as

$$\boldsymbol{u}(\boldsymbol{p}) = \boldsymbol{u}(\boldsymbol{q}) + \boldsymbol{A}\,(\boldsymbol{p} - \boldsymbol{q}), \tag{3.85}$$

for all points $\boldsymbol{p}$ and $\boldsymbol{q}$ with $\boldsymbol{A} \equiv \nabla_p \boldsymbol{u}$ a skew tensor. Alternatively, with $\boldsymbol{a}$ denoting the axial vector of $\boldsymbol{A}$, $\boldsymbol{u}$ can be expressed as

$$\boldsymbol{u}(\boldsymbol{p}) = \boldsymbol{u}(\boldsymbol{q}) + \boldsymbol{a} \times (\boldsymbol{p} - \boldsymbol{q}). \tag{3.86}$$

Any displacement that admits the representation (3.85)–(3.86) is called an *infinitesimal rigid displacement* field. Note that infinitesimal rigid displacements have the same representation as rigid velocity fields (see expressions (3.13) and (3.14)).

3.2.3. INFINITESIMAL ISOCHORIC AND VOLUMETRIC DEFORMATIONS

Analogously to the isochoric/volumetric split of the deformation gradient in the finite strain context (refer to Section 3.1.5), the infinitesimal strain tensor $\boldsymbol{\varepsilon}$ can also be split into a purely volumetric and a volume-preserving contribution. The isochoric/volumetric split of the infinitesimal strain tensor is *additive* (in contrast to the *multiplicative* split of the deformation gradient in the finite strain theory) and reads

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_{\mathrm{d}} + \boldsymbol{\varepsilon}_{\mathrm{v}}, \tag{3.87}$$

where

$$\boldsymbol{\varepsilon}_{\mathrm{d}} \equiv \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_{\mathrm{v}} \tag{3.88}$$

is the isochoric component, known as the *strain deviator* or *deviatoric strain*, which measures pure infinitesimal distortions. The tensor

$$\boldsymbol{\varepsilon}_{\mathrm{v}} \equiv \tfrac{1}{3}\,\varepsilon_{\mathrm{v}}\,\boldsymbol{I} \tag{3.89}$$

is the infinitesimal *volumetric strain tensor*. The scalar invariant of $\boldsymbol{\varepsilon}$, defined as

$$\varepsilon_{\mathrm{v}} \equiv I_1(\boldsymbol{\varepsilon}) = \mathrm{tr}\,\boldsymbol{\varepsilon} = \mathrm{tr}\,\nabla^s \boldsymbol{u} = \mathrm{tr}\,\nabla \boldsymbol{u} \tag{3.90}$$

is named the infinitesimal *volumetric strain*. An infinitesimal deformation is volume-preserving if and only if

$$\varepsilon_{\mathrm{v}} = 0. \tag{3.91}$$

The tensors $\boldsymbol{\varepsilon}_{\mathrm{d}}$ and $\boldsymbol{\varepsilon}_{\mathrm{v}}$ can be equivalently written in terms of linear operations on $\boldsymbol{\varepsilon}$ as

$$\boldsymbol{\varepsilon}_{\mathrm{d}} = [\mathsf{I}_S - \tfrac{1}{3}\,\boldsymbol{I}\otimes\boldsymbol{I}] : \boldsymbol{\varepsilon}, \quad \boldsymbol{\varepsilon}_{\mathrm{v}} = \tfrac{1}{3}\,(\boldsymbol{I}\otimes\boldsymbol{I}) : \boldsymbol{\varepsilon}. \tag{3.92}$$

It should be noted that the strain deviator is a *traceless* tensor, i.e.

$$\mathrm{tr}\,\boldsymbol{\varepsilon}_{\mathrm{d}} = 0. \tag{3.93}$$

The fourth-order tensor defined as

$$\mathsf{I}_{\mathrm{d}} \equiv \mathsf{I}_S - \tfrac{1}{3}\,\boldsymbol{I}\otimes\boldsymbol{I}, \tag{3.94}$$

is referred to as the *deviatoric projection tensor*. It projects second-order symmetric tensors into the *deviatoric subspace*, i.e. into the space of traceless tensors. Throughout this book we shall often use the alternative notation

$$\mathrm{dev}(\boldsymbol{S})$$

to represent the deviator of a symmetric tensor $\boldsymbol{S}$, i.e.

$$\mathrm{dev}(\boldsymbol{S}) \equiv \mathsf{I}_{\mathrm{d}} : \boldsymbol{S}. \tag{3.95}$$

From finite to infinitesimal isochoric and volumetric strains

Analogously to Section 3.2.1, where the infinitesimal strain tensor is derived from the finite strain theory, the above isochoric/volumetric split can also be obtained from its finite deformation counterpart by neglecting higher order terms in $\nabla_p \boldsymbol{u}$.

To show this, let us consider the Green–Lagrange strain tensor, $\boldsymbol{E}^{(2)}$. Following the isochoric/volumetric split of the deformation gradient given by (3.35), we define the corresponding isochoric and volumetric Green–Lagrange strains

$$\boldsymbol{E}^{(2)}_{\mathrm{iso}} \equiv \tfrac{1}{2}(\boldsymbol{C}_{\mathrm{iso}} - \boldsymbol{I}); \quad \boldsymbol{E}^{(2)}_{\mathrm{v}} \equiv \tfrac{1}{2}(\boldsymbol{C}_{\mathrm{v}} - \boldsymbol{I}), \tag{3.96}$$

where

$$\boldsymbol{C}_{\mathrm{iso}} \equiv \boldsymbol{F}^T_{\mathrm{iso}}\,\boldsymbol{F}_{\mathrm{iso}} = (\det\,\boldsymbol{F})^{-\frac{2}{3}}\,\boldsymbol{F}^T\,\boldsymbol{F} = (\det\,\boldsymbol{F})^{-\frac{2}{3}}\,\boldsymbol{C} \tag{3.97}$$

and

$$\boldsymbol{C}_{\mathrm{v}} \equiv \boldsymbol{F}^T_{\mathrm{v}}\,\boldsymbol{F}_{\mathrm{v}} = (\det\,\boldsymbol{F})^{\frac{2}{3}}\,\boldsymbol{I}. \tag{3.98}$$

Now we proceed to show that, under small strain conditions (small $\nabla_p \boldsymbol{u}$), the volumetric Green–Lagrange strain defined above leads to definition (3.90). From $(3.96)_2$ and (3.98), we have

$$\boldsymbol{E}^{(2)}_{\mathrm{v}} = \tfrac{1}{2}\,[(\det\,\boldsymbol{F})^{\frac{2}{3}} - 1]\,\boldsymbol{I}. \tag{3.99}$$

From the standard concepts of differentiation discussed in Section 2.5 together with the definition $\boldsymbol{F} = \boldsymbol{I} + \nabla_p \boldsymbol{u}$ and the expression given in (iii) of page 36 for the derivative of the determinant, we find that

$$\begin{aligned} \det \boldsymbol{F} &= \det(\boldsymbol{I} + \nabla_p \boldsymbol{u}) \\ &= \det \boldsymbol{I} + (\det \boldsymbol{I})\, \boldsymbol{I} : \nabla_p \boldsymbol{u} + o(\nabla_p \boldsymbol{u}) \\ &= 1 + \operatorname{tr} \nabla_p \boldsymbol{u} + o(\nabla_p \boldsymbol{u}) \end{aligned} \tag{3.100}$$

and

$$(\det \boldsymbol{F})^{\frac{2}{3}} = 1 + \tfrac{2}{3} \operatorname{tr} \nabla_p \boldsymbol{u} + o(\nabla_p \boldsymbol{u}). \tag{3.101}$$

With the substitution of the above expression into (3.99), we then obtain

$$\boldsymbol{E}_{\mathrm{v}}^{(2)} = \boldsymbol{\varepsilon}_{\mathrm{v}} + \boldsymbol{o}(\nabla_p \boldsymbol{u}). \tag{3.102}$$

Thus, if higher-order terms are neglected, we have the following approximation

$$\boldsymbol{E}_{\mathrm{v}}^{(2)} \approx \boldsymbol{\varepsilon}_{\mathrm{v}}. \tag{3.103}$$

Following a completely analogous procedure with the isochoric Green–Lagrange strain, we obtain

$$\begin{aligned} \boldsymbol{E}_{\mathrm{iso}}^{(2)} &= \tfrac{1}{2}[(\det \boldsymbol{F})^{-\frac{2}{3}} (\boldsymbol{I} + \nabla_p^T \boldsymbol{u} + \nabla_p \boldsymbol{u} + \nabla_p^T \boldsymbol{u}\, \nabla_p \boldsymbol{u}) - \boldsymbol{I}] \\ &= \tfrac{1}{2}\{[1 - \tfrac{2}{3} \operatorname{tr} \nabla_p \boldsymbol{u} + o(\nabla_p \boldsymbol{u})][\boldsymbol{I} + \nabla_p^T \boldsymbol{u} + \nabla_p \boldsymbol{u} + o(\nabla_p \boldsymbol{u})] - \boldsymbol{I}\} \\ &= \boldsymbol{\varepsilon} - \tfrac{1}{3}(\operatorname{tr} \nabla_p \boldsymbol{u})\boldsymbol{I} + o(\nabla_p \boldsymbol{u}) \\ &= \boldsymbol{\varepsilon}_{\mathrm{d}} + \boldsymbol{o}(\nabla_p \boldsymbol{u}). \end{aligned} \tag{3.104}$$

Thus, to within second-order terms in $\nabla_p \boldsymbol{u}$, we have

$$\boldsymbol{E}_{\mathrm{iso}}^{(2)} \approx \boldsymbol{\varepsilon}_{\mathrm{d}}. \tag{3.105}$$

The infinitesimal limits above are valid for all Lagrangian and Eulerian finite strain measures defined by expressions (3.56) and (3.60).

3.3. Forces. Stress Measures

The previous sections of this chapter have been limited to the mathematical description of the kinematics of deformation. In particular, concepts such as the deformation gradient, rotations and the different strain measures used to quantify internal straining are of utmost importance in the formulation of the mechanical and thermodynamical theory of continua. It should be noted that, thus far, no reference has been made to *forces* and how they are transferred within continuum bodies.

The forces associated with the mechanical description of a body can be classed into three categories:[‡]

[‡] Stress *couples* could also be considered but these are outside the scope of this book and fall within the realm of the so-called *polar* continuum theories (Cosserrat and Cosserrat, 1909; Toupin, 1962; Truesdell and Noll, 1965).

1. *Boundary forces*. Forces applied to the boundary of the body such as those resulting from contact with another body. The dimension of boundary forces is force per unit area.
2. *Body forces*. Forces exerted on the interior of the body. Gravitational and magnetic forces are typical examples of such forces. The dimension of body forces is force per unit mass (or volume).
3. *Internal interactions between adjacent parts of a body*. The dimension of such interactions is force per unit area.

Internal interaction forces arise from the action of one part of the body upon an adjacent part and are transmitted across the surface that separate them. Boundary forces represent interactions between the exterior and the interior of a body and, as internal interactions, are transmitted across a surface (the boundary of the body in this case). Thus, boundary forces and interactions between distinct parts of a body are forces of essentially the same type and will be collectively called *surface forces*. To describe surface forces mathematically, the concept of *stress* as well as the different ways of quantifying it are introduced in this section.

3.3.1. CAUCHY'S AXIOM. THE CAUCHY STRESS VECTOR

Crucial to the description of surface forces is *Cauchy's axiom* stated in what follows. Consider a body $\mathscr{B}$ in an arbitrarily deformed configuration (Figure 3.11). Let $\mathscr{S}$ be an oriented surface of $\mathscr{B}$ with unit normal vector $\boldsymbol{n}$ at a point $\boldsymbol{x}$. Cauchy's axiom states that 'At $\boldsymbol{x}$, the surface force, i.e. the force per unit area, exerted across $\mathscr{S}$ by the material on the side of $\mathscr{S}$ into which $\boldsymbol{n}$ is pointing upon the material on the other side of $\mathscr{S}$ depends on $\mathscr{S}$ only through its normal $\boldsymbol{n}$'. This means that identical forces are transmitted across *any* surfaces with normal $\boldsymbol{n}$ at $\boldsymbol{x}$ (such as surfaces $\mathscr{S}$ and $\mathscr{T}$ in Figure 3.11). This force (per unit area) is called the *Cauchy stress vector* and will be denoted

$$\boldsymbol{t}(\boldsymbol{n}),$$

with dependence on $\boldsymbol{x}$ and time omitted for notational convenience. If $\mathscr{S}$ belongs to the boundary of $\mathscr{B}$, then the Cauchy stress vector represents the contact force exerted by the surrounding environment on $\mathscr{B}$.

3.3.2. THE AXIOM OF MOMENTUM BALANCE

Let $\mathscr{B}$ now be subjected to a system of surface forces, $\boldsymbol{t}(\boldsymbol{x}, \boldsymbol{n})$, and body forces, $\boldsymbol{b}(\boldsymbol{x})$. The spatial field $\boldsymbol{b}(\boldsymbol{x})$ represents force per unit mass acting on the interior of $\mathscr{B}$. The *axiom of momentum balance* asserts that 'For any part $\mathscr{P}$ of the deformed configuration of $\mathscr{B}$, with boundary $\mathscr{S}$, the *balance of linear momentum*,

$$\int_{\mathscr{S}} \boldsymbol{t}(\boldsymbol{n})\, \mathrm{d}a + \int_{\mathscr{P}} \rho\, \boldsymbol{b}\, \mathrm{d}v = \int_{\mathscr{P}} \rho\, \dot{\boldsymbol{v}}\, \mathrm{d}v \tag{3.106}$$

and the *balance of angular momentum*,

$$\int_{\mathscr{S}} \boldsymbol{x} \times \boldsymbol{t}(\boldsymbol{n})\, \mathrm{d}a + \int_{\mathscr{P}} \boldsymbol{x} \times \rho\, \boldsymbol{b}\, \mathrm{d}v = \int_{\mathscr{P}} \boldsymbol{x} \times \rho\, \dot{\boldsymbol{v}}\, \mathrm{d}v \tag{3.107}$$

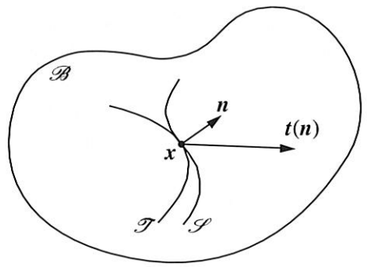

Figure 3.11. Surface forces.

are satisfied, with $\rho = \rho(\boldsymbol{x})$ denoting the *mass density* field, i.e. the mass per unit volume in the deformed configuration of $\mathscr{B}$'. The right-hand sides of (3.106) and (3.107) contain the *inertia* terms, with $\dot{\boldsymbol{v}} = \ddot{\boldsymbol{u}}$ denoting the acceleration field of $\mathscr{B}$.

3.3.3. THE CAUCHY STRESS TENSOR

One of the most fundamental results in continuum mechanics is *Cauchy's theorem* which establishes that, as a consequence of the axiom of momentum balance, the dependence of the surface force $\boldsymbol{t}$ upon the normal $\boldsymbol{n}$ is *linear*, i.e. there exists (recall Section 2.2, starting page 19) a *second-order tensor* field $\boldsymbol{\sigma}(\boldsymbol{x})$ such that the Cauchy stress vector (see Figure 3.12) is given by

$$\boldsymbol{t}(\boldsymbol{x}, \boldsymbol{n}) = \boldsymbol{\sigma}(\boldsymbol{x})\, \boldsymbol{n}. \tag{3.108}$$

Further, $\boldsymbol{\sigma}$ is *symmetric*,[§]

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^T. \tag{3.109}$$

The tensor $\boldsymbol{\sigma}$ is called the *Cauchy stress tensor* and is often referred to as the *true stress tensor* or, simply, *stress tensor*. Formal proofs to Cauchy's theorem can be found, among others, in Wang and Truesdell (1973), Gurtin (1972, 1981), Gurtin and Martins (1976), Marsden and Hughes (1983) and Ciarlet (1988).

At this point, it should be emphasised that, in real life bodies, forces are actually transferred by atomic interactions which are clearly *discrete* quantities. The continuum mathematical representation of such interactions by means of a stress tensor is meaningful only in an average sense and is valid only for a sufficiently large volume of material. This observation applies equally to quantities such as strain measures or any other continuum fields associated with the body. The smallest volume of material for which the continuum representation makes sense is called the *representative volume element*.

Cauchy stress components

Using an orthonormal basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$, the Cauchy stress tensor is represented as

$$\boldsymbol{\sigma} = \sigma_{ij}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j, \tag{3.110}$$

[§]The symmetry of $\boldsymbol{\sigma}$ is a result of the balance of angular momentum.

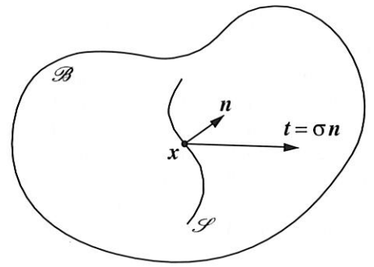

Figure 3.12. The Cauchy stress.

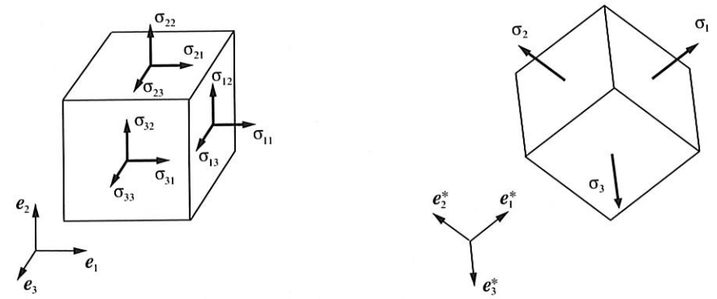

Figure 3.13. Cauchy stress tensor components and principal Cauchy stresses.

with summation on repeated indices implied and the components σ_{ij} given by

$$\sigma_{ij} = (\boldsymbol{\sigma}\, \boldsymbol{e}_i) \cdot \boldsymbol{e}_j. \tag{3.111}$$

From (3.108), it follows that the vector $\boldsymbol{\sigma}\, \boldsymbol{e}_i$ is the force per unit area exerted across a surface whose unit normal vector is $\boldsymbol{e}_i$ at the point of interest. The component σ_{ij} of the Cauchy stress tensor is the magnitude of the projection of $\boldsymbol{\sigma}\, \boldsymbol{e}_i$ in the direction of $\boldsymbol{e}_j$. The schematic representation of such projections is illustrated in Figure 3.13 where an infinitesimal cube with faces normal to the base vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ is considered. The components σ_{11}, σ_{22} and σ_{33} represent the tractions *normal* to the faces of the infinitesimal cube whereas the remaining components, σ_{12}, σ_{13}, σ_{21}, σ_{23}, σ_{31} and σ_{32} are the *shear* tractions acting parallel to the faces.

Principal Cauchy stresses

Due to its symmetry, the Cauchy stress tensor admits the spectral representation

$$\boldsymbol{\sigma} = \sum_{i=1}^{3} \sigma_i\, \boldsymbol{e}_i^* \otimes \boldsymbol{e}_i^*, \tag{3.112}$$

that is, there exists an orthonormal basis $\{e_1^*, e_2^*, e_3^*\}$, for which all shear components of the Cauchy stress tensor vanish and only the normal components may be non-zero. The normal components, σ_i, are the eigenvalues of $\boldsymbol{\sigma}$ and are called the *principal Cauchy stresses*. The directions defined by the basis $\{e_1^*, e_2^*, e_3^*\}$ are named the *principal stress directions*. The schematic representation of the forces acting on the faces of the infinitesimal cube oriented according to the principal stress directions is shown in Figure 3.13. The forces are exclusively normal to the faces of this cube. Note that, analogously to the representation of the stress tensor in terms of principal stresses, the spectral decomposition has been used in Section 3.1 to represent the stretch tensors $\boldsymbol{U}$ and $\boldsymbol{V}$ in terms of principal stretches (see expression (3.51)).

Deviatoric and hydrostatic stresses

It is often convenient, particularly for the purpose of constitutive modelling, to split the stress tensor $\boldsymbol{\sigma}$ into the sum of a *spherical* and a *traceless* component

$$\boldsymbol{\sigma} = \boldsymbol{s} + p\,\boldsymbol{I}, \tag{3.113}$$

where the invariant

$$p \equiv \tfrac{1}{3}\, I_1(\boldsymbol{\sigma}) = \tfrac{1}{3}\,\mathrm{tr}\,\boldsymbol{\sigma} \tag{3.114}$$

is the *hydrostatic pressure* (also referred to as *hydrostatic stress*, *mean stress* or *mean normal pressure*), and

$$\boldsymbol{s} \equiv \boldsymbol{\sigma} - p\,\boldsymbol{I} = \mathsf{I}_{\mathrm{d}} : \boldsymbol{\sigma}, \tag{3.115}$$

with I_d defined by (3.94), is a traceless tensor named the *deviatoric stress* or *stress deviator*. The tensor

$$p\,\boldsymbol{I} = \tfrac{1}{3}(\boldsymbol{I} \otimes \boldsymbol{I}) : \boldsymbol{\sigma} \tag{3.116}$$

is called the *spherical stress tensor*. The above decomposition is analogous to the isochoric/volumetric split of the infinitesimal strain tensor discussed in Section 3.2.3.

3.3.4. THE FIRST PIOLA–KIRCHHOFF STRESS

The traction vector $\boldsymbol{t}$ of expression (3.108) measures the force exerted across a material surface per unit *deformed* area. Crucial to the definition of the first Piola–Kirchhoff stress is the counterpart $\bar{\boldsymbol{t}}$ of $\boldsymbol{t}$ that measures, at the point of interest, the force that acts across any surface whose normal is $\boldsymbol{n}$ in the deformed configuration per unit *reference* area. With $\mathrm{d}a$ denoting an infinitesimal element of area of a surface normal to $\boldsymbol{n}$ in the deformed configuration and with $\mathrm{d}a_0$ being its undeformed counterpart, $\bar{\boldsymbol{t}}$ is expressed as (Figure 3.14)

$$\bar{\boldsymbol{t}} = \frac{\mathrm{d}a}{\mathrm{d}a_0}\,\boldsymbol{t} = \frac{\mathrm{d}a}{\mathrm{d}a_0}\,\boldsymbol{\sigma}\,\boldsymbol{n}. \tag{3.117}$$

Consider the surface $\mathscr{S}$ in the *reference* configuration of $\mathscr{B}$ (Figure 3.14). Let $\mathrm{d}\boldsymbol{p}_1$ and $\mathrm{d}\boldsymbol{p}_2$ be infinitesimal (linearly independent) vectors tangent to $\mathscr{S}$ at the material point $\boldsymbol{p}$ and let $\mathrm{d}a_0$ be the area element generated by $\mathrm{d}\boldsymbol{p}_1$ and $\mathrm{d}\boldsymbol{p}_2$. With $\boldsymbol{m}$ denoting the unit normal to $\mathscr{S}$ at $\boldsymbol{p}$, one has

$$\boldsymbol{m}\,\mathrm{d}a_0 = \mathrm{d}\boldsymbol{p}_1 \times \mathrm{d}\boldsymbol{p}_2. \tag{3.118}$$

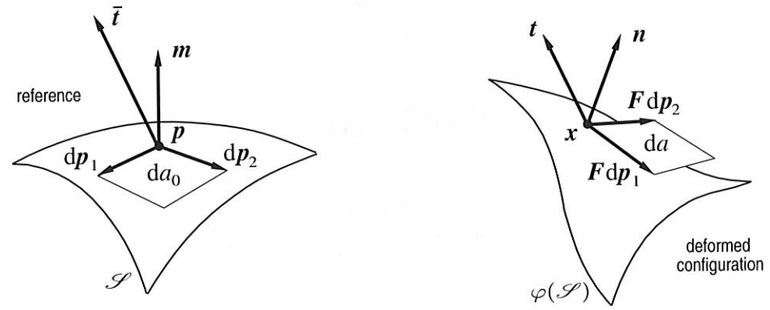

Figure 3.14. The first Piola–Kirchhoff stress tensor.

Under deformation, the tangent vectors $\mathrm{d}\boldsymbol{p}_1$ and $\mathrm{d}\boldsymbol{p}_2$ are mapped, respectively, into $\boldsymbol{F}\,\mathrm{d}\boldsymbol{p}_1$ and $\boldsymbol{F}\,\mathrm{d}\boldsymbol{p}_2$ so that the unit normal to the deformed configuration of $\mathscr{S}$ reads

$$\boldsymbol{n}\,\mathrm{d}a = \boldsymbol{F}\,\mathrm{d}\boldsymbol{p}_1 \times \boldsymbol{F}\,\mathrm{d}\boldsymbol{p}_2, \tag{3.119}$$

where $\mathrm{d}a$ is the corresponding deformed area element. Pre-multiplication of both sides of the expression above by $\boldsymbol{F}^T$ together with use of the identity

$$\boldsymbol{S}\boldsymbol{u} \times \boldsymbol{S}\boldsymbol{v} = (\det \boldsymbol{S})\,\boldsymbol{S}^{-T}\,(\boldsymbol{u} \times \boldsymbol{v}), \tag{3.120}$$

valid for any invertible tensor $\boldsymbol{S}$ and vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, leads to

$$\boldsymbol{F}^T\boldsymbol{n}\,\mathrm{d}a = J\,\mathrm{d}\boldsymbol{p}_1 \times \mathrm{d}\boldsymbol{p}_2 = J\,\boldsymbol{m}\,\mathrm{d}a_0, \tag{3.121}$$

where $J \equiv \det \boldsymbol{F}$. This is equivalent to

$$\frac{\mathrm{d}a}{\mathrm{d}a_0}\,\boldsymbol{n} = J\,\boldsymbol{F}^{-T}\boldsymbol{m}. \tag{3.122}$$

Finally, with substitution of the expression above into (3.117), $\bar{\boldsymbol{t}}$ may be written in terms of the reference unit normal $\boldsymbol{m}$ as

$$\bar{\boldsymbol{t}} = J\,\boldsymbol{\sigma}\,\boldsymbol{F}^{-T}\,\boldsymbol{m}. \tag{3.123}$$

This last expression motivates the following definition

$$\boldsymbol{P} \equiv J\,\boldsymbol{\sigma}\,\boldsymbol{F}^{-T}, \tag{3.124}$$

so that the force transmitted across $\mathscr{S}$ measured per unit *reference* area reads

$$\bar{\boldsymbol{t}} = \boldsymbol{P}\,\boldsymbol{m}. \tag{3.125}$$

The tensor $\boldsymbol{P}$ is called the *first Piola–Kirchhoff stress* and is often referred to as the *Piola–Kirchhoff stress* or *nominal stress*.¶ The vector $\bar{\boldsymbol{t}}$ is obtained by applying the first Piola–Kirchhoff stress to the unit vector $\boldsymbol{m}$, normal to the *reference* configuration of $\mathscr{S}$ at the point of interest. Note that in contrast to the Cauchy stress, $\boldsymbol{P}$ is generally *unsymmetric*.

¶Some authors (Billington and Tate, 1981; Nemat-Nasser, 1999) define the nominal stress as the transpose of the first Piola–Kirchhoff stress tensor.

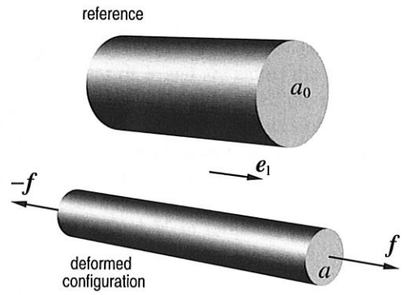

Figure 3.15. The first Piola–Kirchhoff stress. Example.

Example 3.3.1 (The Piola–Kirchhoff stress). Consider a cylindrical bar (Figure 3.15) with cross-sectional area a_0 in its initial configuration (taken as reference). During a uniaxial experiment this bar is stretched along its longitudinal axis (direction of $\boldsymbol{e}_1$) with a simultaneous reduction of its cross section. Assume that the final deformed configuration of the bar corresponds to a state of homogeneous deformation with cross-sectional area a. Furthermore, assume that the bar is subjected to a state of *uniaxial* stress, with constant $\boldsymbol{\sigma}$ given by

$$\boldsymbol{\sigma} = \sigma_{11}\, \boldsymbol{e}_1 \otimes \boldsymbol{e}_1.$$

Let $\boldsymbol{f} = f\, \boldsymbol{e}_1$ be the total force applied to the deformed configuration of the bar (by the experimental equipment). Under the assumption of uniform stress distribution in the cross-section of the bar, force balance requires that the Cauchy stress component σ_{11} be given by

$$\sigma_{11} = \frac{f}{a}.$$

In practice, the force $\boldsymbol{f}$ (and not the stress component) is what can actually be measured in an experiment. Thus, after $\boldsymbol{f}$ is measured, the Cauchy stress σ_{11} is determined according to the expression above. If instead of a, the *reference* cross-sectional area a_0 is used, then the *first Piola–Kirchhoff* or *nominal* stress component is determined

$$P_{11} = \frac{f}{a_0}.$$

It is obvious that, in this case, the corresponding tractions $\boldsymbol{t}$ and $\bar{\boldsymbol{t}}$, respectively per unit deformed and reference area, are simply

$$\boldsymbol{t} = \sigma_{11}\, \boldsymbol{e}_1 = \frac{1}{a}\, \boldsymbol{f}, \quad \bar{\boldsymbol{t}} = P_{11}\, \boldsymbol{e}_1 = \frac{1}{a_0}\, \boldsymbol{f}.$$

3.3.5. THE SECOND PIOLA–KIRCHHOFF STRESS

The *Second Piola–Kirchhoff* stress tensor, denoted $\boldsymbol{S}$, is the tensor defined as

$$\boldsymbol{S} \equiv J \boldsymbol{F}^{-1} \boldsymbol{\sigma} \boldsymbol{F}^{-T}. \tag{3.126}$$

Note that from this definition, we have

$$\boldsymbol{S}^T = J \boldsymbol{F}^{-1} \boldsymbol{\sigma}^T \boldsymbol{F}^{-T}, \tag{3.127}$$

so that the symmetry of $\boldsymbol{\sigma}$ implies that $\boldsymbol{S}$ is symmetric.

3.3.6. THE KIRCHHOFF STRESS

Another important measure of stress is the *Kirchhoff stress tensor*, $\boldsymbol{\tau}$, defined by

$$\boldsymbol{\tau} \equiv J\, \boldsymbol{\sigma}. \tag{3.128}$$

Due to the symmetry of $\boldsymbol{\sigma}$, the Kirchhoff stress is symmetric. Its spectral representation reads

$$\boldsymbol{\tau} = \sum_{i=1}^{3} \tau_i\, \boldsymbol{e}_i^* \otimes \boldsymbol{e}_i^*, \tag{3.129}$$

where the *principal Kirchhoff stresses*, τ_i, are related to the principal Cauchy stresses, σ_i, by

$$\tau_i = J\, \sigma_i. \tag{3.130}$$

Later in this book, frequent reference to the principal Kirchhoff stresses will be made in the formulation of various constitutive models.

3.4. Fundamental laws of thermodynamics

In order to state the fundamental laws of thermodynamics, it is necessary to introduce the scalar fields θ, e, s and r defined over $\mathscr{B}$ which represent, respectively, the *temperature*, *specific internal energy*, *specific entropy* and the *density of heat production*. In addition, $\boldsymbol{b}$ and $\boldsymbol{q}$ will denote the vector fields corresponding, respectively, to the *body force* (force per unit volume in the deformed configuration) and *heat flux*.

3.4.1. CONSERVATION OF MASS

The postulate of conservation of mass requires that

$$\dot{\rho} + \rho\, \mathrm{div}_x\, \dot{\boldsymbol{u}} = 0. \tag{3.131}$$

3.4.2. MOMENTUM BALANCE

In terms of the Cauchy stress tensor, whose existence has been established in Section 3.3.3, the balance of momentum for $\mathscr{B}$ can be expressed by the following partial differential equation with boundary condition:‖

$$\begin{aligned} \mathrm{div}_x\, \boldsymbol{\sigma} + \boldsymbol{b} &= \rho\, \ddot{\boldsymbol{u}} \quad \text{in } \varphi(\Omega) \\ \boldsymbol{t} &= \boldsymbol{\sigma}\, \boldsymbol{n} \quad \text{in } \varphi(\partial\Omega), \end{aligned} \tag{3.132}$$

‖ Equations (3.132) are also a result of Cauchy's theorem, alluded to in page 62.

where $\boldsymbol{n}$ is the outward unit vector normal to the deformed boundary $\boldsymbol{\varphi}(\partial\Omega)$ of $\mathscr{B}$ and $\boldsymbol{t}$ is the applied boundary traction vector field on $\boldsymbol{\varphi}(\partial\Omega)$. Equations (3.132) are often referred to as the *strong*, *local* or *point-wise* form of equilibrium. Equation $(3.132)_1$ is known as *Cauchy's equation of motion*.

The above momentum balance equations are formulated in the *spatial* (deformed) configuration. Equivalently, they may be expressed in the *reference* (or material) configuration of $\mathscr{B}$ in terms of the first Piola–Kirchhoff stress tensor as

$$\begin{aligned} \operatorname{div}_p \boldsymbol{P} + \bar{\boldsymbol{b}} &= \bar{\rho}\,\ddot{\boldsymbol{u}} \quad \text{in } \Omega \\ \bar{\boldsymbol{t}} &= \boldsymbol{P}\,\boldsymbol{m} \qquad \text{in } \partial\Omega, \end{aligned} \tag{3.133}$$

where

$$\bar{\boldsymbol{b}} = J\,\boldsymbol{b} \tag{3.134}$$

is the *reference body force*, i.e. the body force measured per unit volume in the reference configuration,

$$\bar{\rho} = J\,\rho, \tag{3.135}$$

is the *reference density* (mass per unit volume in the reference configuration), $\bar{\boldsymbol{t}}$ is the *reference boundary traction* (boundary force per unit reference area) and $\boldsymbol{m}$ is the outward normal to the boundary of $\mathscr{B}$ in its reference configuration.

3.4.3. THE FIRST PRINCIPLE

The first principle of thermodynamics postulates the *conservation of energy*. Before stating this principle, it is convenient to introduce the product

$$\boldsymbol{\sigma} : \boldsymbol{D},$$

which represents the *stress power* per unit volume in the deformed configuration of a body. The first principle of thermodynamics is mathematically expressed by the equation

$$\rho\,\dot{e} = \boldsymbol{\sigma} : \boldsymbol{D} + \rho\,r - \operatorname{div}_x \boldsymbol{q}. \tag{3.136}$$

In words, the rate of internal energy per unit deformed volume must equal the sum of the stress power and heat production per unit deformed volume minus the spatial divergence of the heat flux.

3.4.4. THE SECOND PRINCIPLE

The second principle of thermodynamics postulates the *irreversibility of entropy production*. It is expressed by means of the inequality

$$\rho\,\dot{s} + \operatorname{div}_x\left[\frac{\boldsymbol{q}}{\theta}\right] - \frac{\rho\,r}{\theta} \geq 0. \tag{3.137}$$

3.4.5. THE CLAUSIUS–DUHEM INEQUALITY

By combination of the first and second principles stated above, one easily obtains the fundamental inequality

$$\rho\,\dot{s} + \operatorname{div}_x\left[\frac{\boldsymbol{q}}{\theta}\right] - \frac{1}{\theta}(\rho\,\dot{e} - \boldsymbol{\sigma} : \boldsymbol{D} + \operatorname{div}_x \boldsymbol{q}) \geq 0. \tag{3.138}$$

The introduction of the *specific free energy* ψ (also known as the *Helmholtz free energy per unit mass*), defined by

$$\psi = e - \theta\,s, \tag{3.139}$$

along with the identity

$$\operatorname{div}_x\left[\frac{\boldsymbol{q}}{\theta}\right] = \frac{1}{\theta}\operatorname{div}_x \boldsymbol{q} - \frac{1}{\theta^2}\,\boldsymbol{q}\cdot\nabla_x\theta, \tag{3.140}$$

into the above fundamental inequality results in the *Clausius–Duhem inequality*

$$\boldsymbol{\sigma} : \boldsymbol{D} - \rho(\dot{\psi} + s\,\dot{\theta}) - \frac{1}{\theta}\,\boldsymbol{q}\cdot\boldsymbol{g} \geq 0, \tag{3.141}$$

where we have defined $\boldsymbol{g} = \nabla_x\theta$. The left-hand side of (3.141) represents the dissipation per unit *deformed* volume. Equivalently, by making use of (3.135), the Clausius–Duhem inequality can be expressed in terms of dissipation per unit *reference* volume as

$$\boldsymbol{\tau} : \boldsymbol{D} - \bar{\rho}(\dot{\psi} + s\,\dot{\theta}) - \frac{J}{\theta}\,\boldsymbol{q}\cdot\boldsymbol{g} \geq 0. \tag{3.142}$$

3.5. Constitutive theory

The balance principles presented so far are valid for any continuum body, regardless of the material of which the body is made. In order to distinguish between different types of material, a *constitutive model* must be introduced. In this section, we review the principles that form the basis of the constitutive theories discussed in later chapters of this book. We start by stating, in Section 3.5.1, three fundamental axioms that define a rather general class of constitutive models of continua. The use of internal variables to formulate constitutive models of dissipative materials is then addressed in Section 3.5.2. We remark that all dissipative constitutive models discussed in Parts Two and Three of this book are based on the internal variable approach. Section 3.5.4 summarises a generic *purely mechanical* internal variable model. The discussion on constitutive theory ends in Section 3.5.5 where the fundamental constitutive initial value problems are stated.

3.5.1. CONSTITUTIVE AXIOMS

In the present context, the axioms stated in this section must be satisfied for any constitutive model. Before going further, it is convenient to introduce the definitions of *thermokinetic* and *calorodynamic* processes (Truesdell, 1969). A *thermokinetic process* of $\mathscr{B}$ is a pair of fields

$$\boldsymbol{\varphi}(\boldsymbol{p}, t) \quad \text{and} \quad \theta(\boldsymbol{p}, t).$$

A set

$$\{\boldsymbol{\sigma}(\boldsymbol{p}, t),\ e(\boldsymbol{p}, t),\ s(\boldsymbol{p}, t),\ r(\boldsymbol{p}, t),\ \boldsymbol{b}(\boldsymbol{p}, t),\ \boldsymbol{q}(\boldsymbol{p}, t)\}$$

of fields satisfying the balance of momentum, the first and the second principles of thermodynamics is called a *calorodynamic process* of $\mathscr{B}$.

Thermodynamic determinism

The basic axiom underlying the constitutive theory discussed here is the *principle of thermodynamically compatible determinism* (Truesdell, 1969). It postulates that 'the history of the thermokinetic process to which a neighbourhood of a point $\boldsymbol{p}$ of $\mathscr{B}$ has been subjected determines a calorodynamic process for $\mathscr{B}$ at $\boldsymbol{p}$'. In particular, we shall be concerned with so-called *simple materials*, for which the *local* history (history at point $\boldsymbol{p}$ only) of $\boldsymbol{F}$, θ and $\boldsymbol{g}$ suffices to determine the history of the thermokinetic process for constitutive purposes. In this case, regarding the body force $\boldsymbol{b}$ and heat supply r as delivered, respectively, by the linear momentum balance $(3.132)_1$ and conservation of energy (3.136) and introducing the specific free energy (3.139), the principle of thermodynamic determinism implies the existence of constitutive functionals $\mathfrak{F}$, $\mathfrak{G}$, $\mathfrak{H}$ and $\mathfrak{I}$ of the histories of $\boldsymbol{F}$, θ and $\boldsymbol{g}$ such that, for a point $\boldsymbol{p}$,

$$\begin{aligned} \boldsymbol{\sigma}(t) &= \mathfrak{F}\,(\boldsymbol{F}^{\,t}, \theta^t, \boldsymbol{g}^t) \\ \psi(t) &= \mathfrak{G}(\boldsymbol{F}^{\,t}, \theta^t, \boldsymbol{g}^t) \\ s(t) &= \mathfrak{H}(\boldsymbol{F}^{\,t}, \theta^t, \boldsymbol{g}^t) \\ \boldsymbol{q}(t) &= \mathfrak{I}\,(\boldsymbol{F}^{\,t}, \theta^t, \boldsymbol{g}^t) \end{aligned} \tag{3.143}$$

and the Clausius–Duhem inequality (3.141) holds for every thermokinetic process of $\mathscr{B}$. The dependence on $\boldsymbol{p}$ is understood on both sides of (3.143) and $(\cdot)^t$ on the right-hand sides denotes the *history* of $(\cdot)$ at $\boldsymbol{p}$ up to time t.

Material objectivity

Another fundamental axiom of the constitutive theory is the *principle of material objectivity* (or *frame invariance*). It states that 'the material response is independent of the observer'. The motion $\boldsymbol{\varphi}^*$ is related to the motion $\boldsymbol{\varphi}$ by a change in observer if it can be expressed as

$$\boldsymbol{\varphi}^*(\boldsymbol{p}, t) = \boldsymbol{y}(t) + \boldsymbol{Q}(t)\,[\boldsymbol{\varphi}(\boldsymbol{p}, t) - \boldsymbol{x}_0], \tag{3.144}$$

where $\boldsymbol{y}(t)$ is a point in space, $\boldsymbol{Q}(t)$ is a rotation and $\boldsymbol{\varphi}(\boldsymbol{p}, t) - \boldsymbol{x}_0$ is the position vector of $\boldsymbol{\varphi}(\boldsymbol{p}, t)$ relative to an arbitrary origin $\boldsymbol{x}_0$. This relation corresponds to a rigid relative movement between the different observers and the deformation gradient corresponding to $\boldsymbol{\varphi}^*$ is given by

$$\boldsymbol{F}^{\,*} = \boldsymbol{Q}\,\boldsymbol{F}. \tag{3.145}$$

Scalar fields (such as θ, ψ and s) are unaffected by a change in observer but the Cauchy stress $\boldsymbol{\sigma}(t)$, heat flux $\boldsymbol{q}(t)$ and the temperature gradient $\boldsymbol{g}(t)$ transform according to the rules

$$\begin{aligned} \boldsymbol{\sigma} &\longrightarrow \boldsymbol{\sigma}^* = \boldsymbol{Q}\,\boldsymbol{\sigma}\,\boldsymbol{Q}^T \\ \boldsymbol{q} &\longrightarrow \boldsymbol{q}^* = \boldsymbol{Q}\,\boldsymbol{q} \\ \boldsymbol{g} &\longrightarrow \boldsymbol{g}^* = \boldsymbol{Q}\,\boldsymbol{g}. \end{aligned} \tag{3.146}$$

The principle of material objectivity places restrictions on the constitutive functionals (3.143). Formally, it requires that $\mathfrak{F}$, $\mathfrak{G}$, $\mathfrak{H}$ and $\mathfrak{I}$ satisfy

$$\begin{aligned} \boldsymbol{\sigma}^*(t) &= \mathfrak{F}\,(\boldsymbol{F}^{\,t*}, \theta^t, \boldsymbol{g}^{t*}) \\ \psi(t) &= \mathfrak{G}(\boldsymbol{F}^{\,t*}, \theta^t, \boldsymbol{g}^{t*}) \\ s(t) &= \mathfrak{H}(\boldsymbol{F}^{\,t*}, \theta^t, \boldsymbol{g}^{t*}) \\ \boldsymbol{q}^*(t) &= \mathfrak{I}\,(\boldsymbol{F}^{\,t*}, \theta^t, \boldsymbol{g}^{t*}) \end{aligned} \tag{3.147}$$

for any transformation of the form (3.145, 3.146).

Material symmetry

The *symmetry group* of a material is the set of density preserving changes of reference configuration under which the material response functionals $\mathfrak{F}$, $\mathfrak{G}$, $\mathfrak{H}$ and $\mathfrak{I}$ are not affected. The symmetry group of a *solid* material is a subset of the proper orthogonal group $\mathscr{O}^+$, that is, a set of *rotations*. Thus, the symmetry group of a solid material is the set of rotations of the reference configuration under which the response functionals remain unchanged. This concept is expressed mathematically as follows. A subgroup $\mathscr{S}$ of $\mathscr{O}^+$ is said to be the symmetry group of the material defined by the constitutive functionals $\mathfrak{F}$, $\mathfrak{G}$, $\mathfrak{H}$ and $\mathfrak{I}$ if the relations

$$\begin{aligned} \mathfrak{F}\,(\boldsymbol{F}^{\,t}, \theta^t, \boldsymbol{g}^t) &= \mathfrak{F}\,([\boldsymbol{F}\,\boldsymbol{Q}]^t, \theta^t, \boldsymbol{g}^t) \\ \mathfrak{G}\,(\boldsymbol{F}^{\,t}, \theta^t, \boldsymbol{g}^t) &= \mathfrak{G}([\boldsymbol{F}\,\boldsymbol{Q}]^t, \theta^t, \boldsymbol{g}^t) \\ \mathfrak{H}\,(\boldsymbol{F}^{\,t}, \theta^t, \boldsymbol{g}^t) &= \mathfrak{H}([\boldsymbol{F}\,\boldsymbol{Q}]^t, \theta^t, \boldsymbol{g}^t) \\ \mathfrak{I}\,(\boldsymbol{F}^{\,t}, \theta^t, \boldsymbol{g}^t) &= \mathfrak{I}\,([\boldsymbol{F}\,\boldsymbol{Q}]^t, \theta^t, \boldsymbol{g}^t) \end{aligned} \tag{3.148}$$

hold for any time-*independent* rotation $\boldsymbol{Q} \in \mathscr{S}$. A solid is said to be *isotropic*** if its symmetry group is the entire proper orthogonal group. In the development of any constitutive model, the constitutive functionals must comply with the restrictions imposed by the symmetries of the material in question.

3.5.2. THERMODYNAMICS WITH INTERNAL VARIABLES

The constitutive equations (3.143) written in terms of functionals of the history of $\boldsymbol{F}$, θ and $\boldsymbol{g}$, in that format, are far too general to have practical utility in modelling real materials undergoing real thermodynamical process. This is especially true if one has in mind the experimental identification of the constitutive functionals and the solution of the boundary value problems of practical interest. Therefore, it is imperative that simplifying assumptions are added to the general forms of the constitutive relations stated above.

An effective alternative to the general description based on history functionals is the adoption of the so-called *thermodynamics with internal variables*. The starting point of the thermodynamics with internal variables is the hypothesis that at any instant of a thermodynamical process the thermodynamic state (defined by $\boldsymbol{\sigma}$, ψ, s and $\boldsymbol{q}$) at a given

**We remark that most constitutive models discussed in this book are isotropic.

point $\boldsymbol{p}$ can be completely determined by the knowledge of a finite number of *state variables*. The thermodynamic state depends only on the *instantaneous* value of the state variables and not on their past history.

Mathematically, state variable models can be seen as particular instances of the general history functional-based constitutive theory. The relationship between the two approaches is discussed in detail by Kestin and Bataille (1977) and Bataille and Kestin (1979). In general terms, state variable models can be obtained from the general history functional-based description by re-defining the history of the thermokinetic process in terms of a finite number of parameters (the state variables).

The state variables

For the applications with which we are mostly concerned, it will be convenient to assume that at any time t, the thermodynamic state at a point is determined by the following set of *state variables*:

$$\{\boldsymbol{F},\ \theta,\ \boldsymbol{g},\ \boldsymbol{\alpha}\},$$

where $\boldsymbol{F}$, θ and $\boldsymbol{g}$ are the *instantaneous* values of deformation gradient, temperature and the temperature gradient and

$$\boldsymbol{\alpha} = \{\alpha_k\} \tag{3.149}$$

is a set of *internal variables* containing, in general, entities of scalar, vectorial and tensorial nature associated with dissipative mechanisms.

Thermodynamic potential. Stress constitutive equation

Following the above hypothesis, the specific free energy is assumed to have the form[††]

$$\psi = \psi(\boldsymbol{F},\ \theta,\ \boldsymbol{\alpha}), \tag{3.150}$$

so that its rate of change is given by

$$\dot{\psi} = \frac{\partial \psi}{\partial \boldsymbol{F}} : \dot{\boldsymbol{F}} + \frac{\partial \psi}{\partial \theta}\,\dot{\theta} + \frac{\partial \psi}{\partial \alpha_k}\,\dot{\alpha}_k, \tag{3.151}$$

where summation over k is implied. In this case, using the connection

$$\boldsymbol{\sigma} : \boldsymbol{D} = \boldsymbol{\sigma}\,\boldsymbol{F}^{-T} : \dot{\boldsymbol{F}}\,, \tag{3.152}$$

for the stress power, one obtains for the Clausius–Duhem inequality

$$\left(\boldsymbol{\sigma}\,\boldsymbol{F}^{-T} - \rho\,\frac{\partial \psi}{\partial \boldsymbol{F}}\right) : \dot{\boldsymbol{F}} - \rho\left(s + \frac{\partial \psi}{\partial \theta}\right)\dot{\theta} - \rho\,\frac{\partial \psi}{\partial \alpha_k}\,\dot{\alpha}_k - \frac{1}{\theta}\,\boldsymbol{q}\cdot\boldsymbol{g} \geq 0. \tag{3.153}$$

Equivalently, in terms of power per unit *reference* volume, we have

$$\left(\boldsymbol{P} - \bar{\rho}\,\frac{\partial \psi}{\partial \boldsymbol{F}}\right) : \dot{\boldsymbol{F}} - \bar{\rho}\left(s + \frac{\partial \psi}{\partial \theta}\right)\dot{\theta} - \bar{\rho}\,\frac{\partial \psi}{\partial \alpha_k}\,\dot{\alpha}_k - \frac{J}{\theta}\,\boldsymbol{q}\cdot\boldsymbol{g} \geq 0. \tag{3.154}$$

[††]The dependence of ψ on the temperature gradient is disregarded as it contradicts the second principle of thermodynamics (Coleman and Gurtin, 1967).

Expression (3.154) is obtained from (3.153), by simply using relation (3.135).

The principle of thermodynamic determinism requires that the constitutive equations must be such that the above inequality holds for any thermokinetic process. Thus, (3.154) must remain valid for any pair of functions $\{\dot{\boldsymbol{F}}(t), \dot{\theta}(t)\}$. This implies the constitutive equations

$$\boldsymbol{P} = \bar{\rho}\,\frac{\partial \psi}{\partial \boldsymbol{F}}, \quad s = -\frac{\partial \psi}{\partial \theta}, \tag{3.155}$$

for the first Piola–Kirchhoff stress and entropy. Equation $(3.155)_1$ is equivalent to the following constitutive relations for the Cauchy and Kirchoff stress tensors:

$$\boldsymbol{\sigma} = \frac{1}{J}\,\bar{\rho}\,\frac{\partial \psi}{\partial \boldsymbol{F}}\,\boldsymbol{F}^{T}, \quad \boldsymbol{\tau} = \bar{\rho}\,\frac{\partial \psi}{\partial \boldsymbol{F}}\,\boldsymbol{F}^{T}. \tag{3.156}$$

Thermodynamical forces

For each internal variable α_k of the set $\boldsymbol{\alpha}$, we define the *conjugate thermodynamical force*

$$A_k \equiv \bar{\rho}\,\frac{\partial \psi}{\partial \alpha_k}. \tag{3.157}$$

With this definition and the identities (3.155), the Clausius–Duhem inequality can be rewritten as

$$-A_k * \dot{\alpha}_k - \frac{J}{\theta}\,\boldsymbol{q}\cdot\boldsymbol{g} \geq 0, \tag{3.158}$$

where we recall that the symbol '$*$' denotes the appropriate product operation between A_k and $\dot{\alpha}_k$. In what follows, we will adopt for convenience the notation

$$\boldsymbol{A} \equiv \{A_k\} \tag{3.159}$$

for the set of thermodynamical forces, so that (3.158) can be expressed in a more compact form as

$$-\boldsymbol{A} * \dot{\boldsymbol{\alpha}} - \frac{J}{\theta}\,\boldsymbol{q}\cdot\boldsymbol{g} \geq 0. \tag{3.160}$$

Dissipation. Evolution of the internal variables

In order to completely characterise a constitutive model, complementary laws associated with the dissipative mechanisms are required. Namely, constitutive equations for the flux variables $\frac{1}{\theta}\boldsymbol{q}$ and $\dot{\boldsymbol{\alpha}}$ must be postulated. In the general case, we assume that the flux variables are given functions of the state variables. The following constitutive equations are then postulated

$$\begin{aligned} \dot{\boldsymbol{\alpha}} &= f(\boldsymbol{F}, \theta, \boldsymbol{g}, \boldsymbol{\alpha}) \\ \frac{1}{\theta}\boldsymbol{q} &= h(\boldsymbol{F}, \theta, \boldsymbol{g}, \boldsymbol{\alpha}). \end{aligned} \tag{3.161}$$

Recalling the principle of thermodynamic determinism, the Clausius–Duhem inequality, now expressed by (3.158), must hold for any process. This requirement places restrictions on the possible forms of the general constitutive functions f and h in (3.161) (the reader is referred to Coleman and Gurtin, 1967; Truesdell, 1969, for further details on this issue). It is also

important to mention that when internal variables of vectorial or tensorial nature are present, it is frequently convenient to re-formulate $(3.161)_1$ in terms of so-called *objective rates* rather than the standard material time derivative of $\boldsymbol{\alpha}$. Objective rates are insensitive to rigid-body motions and may be essential in the definition of a *frame invariant* evolution law for variables representing physical states associated with material directions. Objective rates are discussed in Section 14.10 (starting page 615) in the context of the hypoelastic-based formulation of plasticity models.

Dissipation potential. Normal dissipativity

An effective way of ensuring that (3.158) is satisfied consists in postulating the existence of a scalar-valued *dissipation potential* of the form

$$\Xi = \Xi(\boldsymbol{A}, \boldsymbol{g}; \boldsymbol{F}, \theta, \boldsymbol{\alpha}), \tag{3.162}$$

where the state variables $\boldsymbol{F}$, θ and $\boldsymbol{\alpha}$ appear as parameters. The potential Ξ is assumed convex with respect to each A_k and $\boldsymbol{g}$, non-negative and zero valued at the origin, $\{\boldsymbol{A}, \boldsymbol{g}\} = \{\boldsymbol{0}, \boldsymbol{0}\}$. In addition, the hypothesis of *normal dissipativity* is introduced, i.e. the flux variables are assumed to be determined by the laws

$$\dot{\alpha}_k = -\frac{\partial \Xi}{\partial A_k}, \quad \frac{1}{\theta}\boldsymbol{q} = -\frac{\partial \Xi}{\partial \boldsymbol{g}}. \tag{3.163}$$

A constitutive model defined by (3.150), (3.155) and (3.163) satisfies *a priori* the dissipation inequality. It should be noted, however, that the constitutive description by means of convex potentials as described above is *not* a consequence of thermodynamics but, rather, a convenient tool for formulating constitutive equations without violating thermodynamics. Examples of constitutive models supported by experimental evidence which do not admit representation by means of dissipation potentials are discussed by Onat and Leckie (1988).

3.5.3. PHENOMENOLOGICAL AND MICROMECHANICAL APPROACHES

The success of a constitutive model intended to describe the behaviour of a particular material depends critically on the choice of an appropriate set of internal variables. Since no plausible model will be general enough to describe the response of a material under all processes, we should have in mind that the choice of internal variables must be guided not only by the specific material in question but, equally importantly, by the processes (i.e. the range of thermokinetic processes defined by strain and temperature histories as well as the time span) under which the model is meant to describe the behaviour of the material. The importance of considering the possible thermokinetic processes when devising a constitutive model can be clearly illustrated, for instance, by considering a simple steel bar. When subjected to a sufficiently small axial strain at room temperature, the bar exhibits a behaviour that can be accurately modelled by linear elasticity theory (generalised Hooke's law). If strains become larger, however, linear elasticity may no longer capture the observed response satisfactorily. In this case, a plasticity theory may be more appropriate. With further increase in complexity of the strain history (by introducing, say, cyclic extension), other phenomena such as internal damaging and possibly fracturing may take place and more

refined constitutive models, incorporating a larger number of state variables, will be required. Due account of the possible temperature histories and time span to be considered is also fundamental. At higher temperatures, the long-term behaviour of the steel bar subjected to even a very small strain, may no longer be accurately modelled by the linear elasticity theory. In this case the introduction of time-dependent effects (creep/relaxation) may be essential to produce an acceptable model. In an extreme situation, if the temperature rises above melting point, the bar will cease to be a solid. Under such circumstances, a fluid mechanics theory will be needed to describe the behaviour of the material.

In general, due to the difficulty involved in the identification of the underlying dissipative mechanisms, the choice of the appropriate set of internal variables is somewhat subtle and tends to be biased by the preferences and background of the investigator. In simplistic terms, we may say that constitutive modelling by means of internal variables relies either on a micromechanical or on a phenomenological approach. The micromechanical approach involves the determination of mechanisms and related variables at the atomic, molecular or crystalline levels. In general, these variables are discrete quantities and their continuum (macroscopic) counterparts can be defined by means of homogenisation techniques. The phenomenological approach, on the other hand, is based on the study of the response of the *representative volume element*, i.e. the element of matter large enough to be regarded as a homogeneous continuum. The internal variables in this case will be directly associated with the dissipative behaviour observed at the *macroscopic* level in terms of continuum quantities (such as strain, temperature, etc.). Despite the macroscopic nature of theories derived on the basis of the phenomenological methodology, it should be expected that 'good' phenomenological internal variables will be somehow related to the underlying microscopic dissipation mechanisms.

The phenomenological approach to irreversible thermodynamics has been particularly successful in the field of solid mechanics. Numerous well-established models of solids, such as classical isotropic elastoplasticity and viscoplasticity, discussed in Parts Two and Three of this book, have been developed on a purely phenomenological basis providing evidence of how powerful such an approach to irreversible thermodynamics can be when the major concern is the description of the essentially macroscopic behaviour. In some instances, however, the inclusion of microscopic information becomes essential and a purely phenomenological methodology is unlikely to describe the behaviour of the material with sufficient accuracy. One such case is illustrated in Chapter 16, where a microscopically-based continuum model of ductile metallic crystals is described.

3.5.4. THE PURELY MECHANICAL THEORY

Thermal effects are ignored in the constitutive theories addressed in Parts Two and Three of this book. It is, therefore, convenient at this point to summarise the general internal variable-based constitutive equations in the purely mechanical case. By removing the thermally-related terms of the above theory, we end up with the following set of mechanical constitutive equations:

$$\begin{cases} \psi = \psi(\boldsymbol{F}, \boldsymbol{\alpha}) \\ \boldsymbol{P} = \bar{\rho}\,\dfrac{\partial \psi}{\partial \boldsymbol{F}} \\ \dot{\boldsymbol{\alpha}} = f(\boldsymbol{F}, \boldsymbol{\alpha}). \end{cases} \tag{3.164}$$

The infinitesimal strain case

In *the infinitesimal strain case*, the infinitesimal strain tensor, $\boldsymbol{\varepsilon}$, replaces the deformation gradient and the stress tensor $\boldsymbol{\sigma}$ of the infinitesimal theory replaces the first Piola–Kirchhoff stress. We then have the general constitutive law

$$\begin{cases} \psi = \psi(\boldsymbol{\varepsilon}, \boldsymbol{\alpha}) \\ \boldsymbol{\sigma} = \bar{\rho}\, \dfrac{\partial \psi}{\partial \boldsymbol{\varepsilon}} \\ \dot{\boldsymbol{\alpha}} = f(\boldsymbol{\varepsilon}, \boldsymbol{\alpha}). \end{cases} \tag{3.165}$$

3.5.5. THE CONSTITUTIVE INITIAL VALUE PROBLEM

Our basic constitutive problem is defined as follows: '*Given the history of the deformation gradient (and the history of temperature and temperature gradient, if thermal effects are considered), find the free-energy and stress (plus entropy and heat flux, in the thermomechanical case) according to the constitutive law conceptually expressed by (3.143)*'. If the internal variable approach is adopted in the formulation of the constitutive equations, the generic constitutive problem reduces to the following fundamental mechanical *initial value problem*.

Problem 3.1 (The mechanical constitutive initial value problem). *Given the initial values of the internal variables $\boldsymbol{\alpha}(t_0)$ and the history of the deformation gradient,*

$$\boldsymbol{F}(t), \quad t \in [t_0, T],$$

find the functions $\boldsymbol{P}(t)$ and $\boldsymbol{\alpha}(t)$, for the first Piola–Kirchhoff stress and the set of internal variables, such that the constitutive equations

$$\begin{cases} \boldsymbol{P}(t) = \bar{\rho}\, \dfrac{\partial \psi}{\partial \boldsymbol{F}}\bigg|_t \\ \dot{\boldsymbol{\alpha}}(t) = f(\boldsymbol{F}(t), \boldsymbol{\alpha}(t)) \end{cases} \tag{3.166}$$

are satisfied for every $t \in [t_0, T]$.

In the infinitesimal case, $\boldsymbol{P}$ and $\boldsymbol{F}$ are replaced with $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$, respectively, in the above initial value problem. For completeness, the infinitesimal constitutive initial value problem is stated in the following.

Problem 3.2 (The infinitesimal constitutive initial value problem). *Given the initial values of the internal variables $\boldsymbol{\alpha}(t_0)$ and the history of the infinitesimal strain tensor,*

$$\boldsymbol{\varepsilon}(t), \quad t \in [t_0, T],$$

find the functions $\boldsymbol{\sigma}(t)$ and $\boldsymbol{\alpha}(t)$, for the stress tensor and the set of internal variables, such that the constitutive equations

$$\begin{cases} \boldsymbol{\sigma}(t) = \bar{\rho}\, \dfrac{\partial \psi}{\partial \boldsymbol{\varepsilon}}\bigg|_t \\ \dot{\boldsymbol{\alpha}}(t) = f(\boldsymbol{\varepsilon}(t), \boldsymbol{\alpha}(t)) \end{cases} \tag{3.167}$$

are satisfied for every $t \in [t_0, T]$.

3.6. Weak equilibrium. The principle of virtual work

The *strong* (point-wise, local or differential) forms of the momentum balance have been stated in Section 3.4 by expressions (3.132) and (3.133). In this section, we state the momentum balance equations in their corresponding *weak* (global or integral) forms. The weak equilibrium statement – the *Principle of Virtual Work* – is fundamental to the definition of the basic initial boundary value problem stated in Section 3.7 and, as we shall see in Chapter 4, is the starting point of displacement-based finite element procedures for the analysis of solids.

Again, let us consider the body $\mathscr{B}$ which occupies the region $\Omega \subset \mathscr{E}$ with boundary $\partial\Omega$ in its reference configuration subjected to body forces in its interior and surface tractions on its boundary. In its deformed configuration, $\mathscr{B}$ occupies the region $\boldsymbol{\varphi}(\Omega)$ with boundary $\boldsymbol{\varphi}(\partial\Omega)$ defined through the deformation map $\boldsymbol{\varphi}$.

3.6.1. THE SPATIAL VERSION

The *spatial* version of the principle of virtual work states that the body $\mathscr{B}$ is in *equilibrium* if, and only if, its Cauchy stress field, $\boldsymbol{\sigma}$, satisfies the equation

$$\int_{\varphi(\Omega)} [\boldsymbol{\sigma} : \nabla_x \boldsymbol{\eta} - (\boldsymbol{b} - \rho\, \ddot{\boldsymbol{u}}) \cdot \boldsymbol{\eta}]\, \mathrm{d}v - \int_{\varphi(\partial\Omega)} \boldsymbol{t} \cdot \boldsymbol{\eta}\, \mathrm{d}a = 0, \quad \forall\, \boldsymbol{\eta} \in \mathscr{V}, \tag{3.168}$$

where $\boldsymbol{b}$ and $\boldsymbol{t}$ are the body force per unit deformed volume and boundary traction per unit deformed area and $\mathscr{V}$ is the *space of virtual displacements* of $\mathscr{B}$, i.e. the space of sufficiently regular arbitrary displacements

$$\boldsymbol{\eta} : \boldsymbol{\varphi}(\Omega) \to \mathscr{U}.$$

Equivalence between strong and weak equilibrium statements

When the stress field $\boldsymbol{\sigma}$ is sufficiently smooth, the virtual work equation is equivalent to the strong momentum balance equations (3.132). To show this, let us start by assuming that the field $\boldsymbol{\sigma}$ is sufficiently regular so that we can use the identity (v) of Section 2.5.8 (page 38) to obtain

$$\boldsymbol{\sigma} : \nabla_x \boldsymbol{\eta} = \mathrm{div}_x(\boldsymbol{\sigma}\, \boldsymbol{\eta}) - (\mathrm{div}_x \boldsymbol{\sigma}) \cdot \boldsymbol{\eta}. \tag{3.169}$$

In obtaining the above identity we have used the fact that $\boldsymbol{\sigma}$ is symmetric. Next, by substituting the above expression into (3.168), it follows that

$$\int_{\varphi(\Omega)} [\mathrm{div}_x(\boldsymbol{\sigma}\, \boldsymbol{\eta}) - (\mathrm{div}_x\, \boldsymbol{\sigma} + \boldsymbol{b} - \rho \ddot{\boldsymbol{u}}) \cdot \boldsymbol{\eta}]\, \mathrm{d}v - \int_{\varphi(\partial\Omega)} \boldsymbol{t} \cdot \boldsymbol{\eta}\, \mathrm{d}a = 0, \quad \forall\, \boldsymbol{\eta} \in \mathscr{V}. \tag{3.170}$$

We now concentrate on the first term within the square brackets of the above equation. The divergence theorem (expression $(2.148)_2$, page 37) implies the following identity

$$\int_{\varphi(\Omega)} \mathrm{div}_x(\boldsymbol{\sigma}\, \boldsymbol{\eta})\, \mathrm{d}v = \int_{\varphi(\partial\Omega)} \boldsymbol{\sigma}\, \boldsymbol{\eta} \cdot \boldsymbol{n}\, \mathrm{d}a. \tag{3.171}$$

By taking into account the symmetry of $\boldsymbol{\sigma}$, which implies $\boldsymbol{\sigma}\boldsymbol{\eta} \cdot \boldsymbol{n} = \boldsymbol{\sigma}\boldsymbol{n} \cdot \boldsymbol{\eta}$, together with the above identity, equation (3.170) can be rewritten in the equivalent form

$$\int_{\varphi(\Omega)} (\mathrm{div}_x\, \boldsymbol{\sigma} + \boldsymbol{b} - \rho \ddot{\boldsymbol{u}}) \cdot \boldsymbol{\eta}\, \mathrm{d}v + \int_{\varphi(\partial\Omega)} (\boldsymbol{t} - \boldsymbol{\sigma}\, \boldsymbol{n}) \cdot \boldsymbol{\eta}\, \mathrm{d}a = 0, \quad \forall\, \boldsymbol{\eta} \in \mathscr{V}. \tag{3.172}$$

Finally, since this equation holds for all virtual displacement fields $\boldsymbol{\eta}$, then it follows from the fundamental theorem of variational calculus (refer, for instance, to Gurtin 1972; Oden 1979 or Reddy 1998) that each bracketed term of the above equation must vanish pointwise within their respective domains, i.e. we recover the strong equilibrium equations (3.132).

Conversely, the strong form yields the weak form of equilibrium. This can be shown in a relatively straightforward manner by applying a weighted residual method to the strong form together with use of the divergence theorem.

3.6.2. THE MATERIAL VERSION

The virtual work equation can be equivalently expressed in the *reference* configuration of $\mathscr{B}$. The corresponding *material* (or *reference*) version of the Principle of Virtual Work states that $\mathscr{B}$ is in equilibrium if and only if its first Piola–Kirchhoff stress field, $\boldsymbol{P}$, satisfies

$$\int_{\Omega} [\boldsymbol{P} : \nabla_p \boldsymbol{\eta} - (\bar{\boldsymbol{b}} - \bar{\rho}\ddot{\boldsymbol{u}}) \cdot \boldsymbol{\eta}] \, \mathrm{d}v - \int_{\partial\Omega} \bar{\boldsymbol{t}} \cdot \boldsymbol{\eta} \, \mathrm{d}a = 0, \quad \forall \, \boldsymbol{\eta} \in \mathscr{V}, \tag{3.173}$$

where $\bar{\boldsymbol{b}}$ and $\bar{\boldsymbol{t}}$ are, respectively, the body force per unit reference volume and the surface traction per unit reference area and $\bar{\rho}$ is the mass density in the reference configuration. The space of virtual displacements, $\mathscr{V}$, is accordingly defined as the space of sufficiently regular arbitrary displacement fields

$$\boldsymbol{\eta} : \Omega \to \mathscr{U}.$$

The material version of the virtual work equation is obtained by introducing, in its spatial counterpart, the identities

$$\boldsymbol{\sigma} = \frac{1}{J} \boldsymbol{P} \, \boldsymbol{F}^T; \quad \nabla_x \boldsymbol{a} = \nabla_p \boldsymbol{a} \, \boldsymbol{F}^{-1}, \tag{3.174}$$

where the second expression holds for a generic vector field $\boldsymbol{a}$, and making use of the standard relation (Gurtin, 1981)

$$\int_{\varphi(\Omega)} a(\boldsymbol{x}) \, \mathrm{d}v = \int_{\Omega} J(\boldsymbol{p}) \, a(\boldsymbol{\varphi}(\boldsymbol{p})) \, \mathrm{d}v, \tag{3.175}$$

valid for any scalar field a.

The proof of equivalence between (3.173) and the strong form (3.133) under conditions of sufficient regularity is then analogous to that given for the spatial version discussed in Section 3.6.1 above.

3.6.3. THE INFINITESIMAL CASE

Under infinitesimal deformations, reference and deformed configurations coincide and the virtual work equation reads simply

$$\int_{\Omega} [\boldsymbol{\sigma} : \nabla \boldsymbol{\eta} - (\boldsymbol{b} - \rho \ddot{\boldsymbol{u}}) \cdot \boldsymbol{\eta}] \, dv - \int_{\partial\Omega} \boldsymbol{t} \cdot \boldsymbol{\eta} \, \mathrm{d}a = 0, \quad \forall \, \boldsymbol{\eta} \in \mathscr{V}. \tag{3.176}$$

3.7. The quasi-static initial boundary value problem

Having defined the generic constitutive initial value problems in Section 3.5 and the weak equilibrium statements in Section 3.6, we are now in a position to state the weak form of fundamental initial boundary value problems, whose numerical solution by the finite element method is the main subject of the subsequent chapters of this book. The problems formulated here are restricted to *quasi-static* conditions, where inertia effects are ignored. This is the case on which the numerical methods discussed in this book are focused.

3.7.1. FINITE DEFORMATIONS

Let the body $\mathscr{B}$ (Figure 3.16) be subjected to a prescribed history of body forces

$$\boldsymbol{b}(t), \quad t \in [t_0, T]$$

in its interior. In the above, dependence of $\boldsymbol{b}$ on $\boldsymbol{x}$ is implicitly assumed. In addition, the following boundary conditions are imposed.

(i) *The natural boundary condition*. The history of the surface traction

$$\boldsymbol{t}(t), \quad t \in [t_0, T],$$

with dependence on $\boldsymbol{x}$ implied, is prescribed over the portion of the boundary of $\mathscr{B}$ that occupies the region $\partial\Omega_t$ in its reference configuration.

(ii) *The essential boundary condition*. The *motion* is a prescribed function on the part of the boundary of $\mathscr{B}$ that occupies the region $\partial\Omega_u$ in the reference configuration

$$\bar{\boldsymbol{\varphi}}(\boldsymbol{p}, t) = \boldsymbol{p} + \bar{\boldsymbol{u}}(\boldsymbol{p}, t) \quad t \in [t_0, T], \; \boldsymbol{p} \in \partial\Omega_u,$$

where $\bar{\boldsymbol{u}}$ is the corresponding prescribed boundary displacement field. For simplicity, it is assumed here that $\partial\Omega_u \bigcap \partial\Omega_t = \emptyset$. We define the set of *kinematically admissible displacements* of $\mathscr{B}$ as the set of all sufficiently regular displacement functions that satisfy the kinematic constraint (the essential boundary condition)

$$\mathscr{K} = \{\boldsymbol{u} : \Omega \times \mathscr{R} \to \mathscr{U} | \; \boldsymbol{u}(\boldsymbol{p}, t) = \bar{\boldsymbol{u}}(\boldsymbol{p}, t), \; t \in [t_0, T], \; \boldsymbol{p} \in \partial\Omega_u\}. \tag{3.177}$$

The body $\mathscr{B}$ is assumed to be made from a generic material modelled by the internal variable-based constitutive equations associated with Problem 3.1 (page 76) and the internal variable field, $\boldsymbol{\alpha}$, is known at the initial time t_0, i.e.

$$\boldsymbol{\alpha}(\boldsymbol{p}, t_0) = \boldsymbol{\alpha}_0(\boldsymbol{p}). \tag{3.178}$$

The fundamental quasi-static initial boundary value problem is stated in its spatial version in the following.

Problem 3.3 (The spatial quasi-static initial boundary value problem). *Find a kinematically admissible displacement function, $\boldsymbol{u} \in \mathscr{K}$, such that, for all $t \in [t_0, T]$, the virtual work*

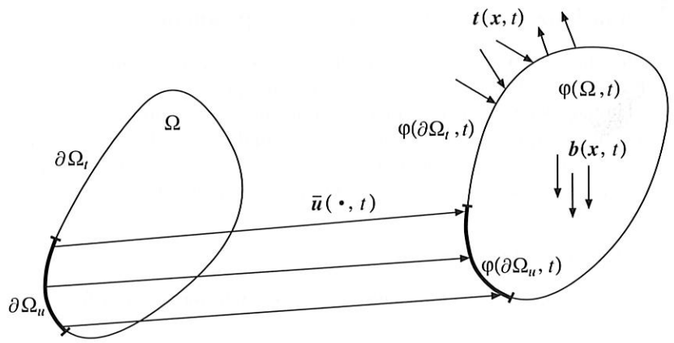

Figure 3.16. The initial boundary value problem. Schematic illustration.

equation is satisfied

$$\int_{\varphi(\Omega,t)} [\boldsymbol{\sigma}(t) : \nabla_x \boldsymbol{\eta} - \boldsymbol{b}(t) \cdot \boldsymbol{\eta}] \, \mathrm{d}v - \int_{\varphi(\partial\Omega_t,t)} \boldsymbol{t}(t) \cdot \boldsymbol{\eta} \, \mathrm{d}a = 0, \quad \forall\, \boldsymbol{\eta} \in \mathscr{V}_t. \tag{3.179}$$

The space of virtual displacements at time t is defined by

$$\mathscr{V}_t = \{\boldsymbol{\eta} : \boldsymbol{\varphi}(\Omega, t) \to \mathscr{U} \mid \boldsymbol{\eta} = \mathbf{0} \text{ on } \boldsymbol{\varphi}(\partial\Omega_u, t)\} \tag{3.180}$$

and, at each point of $\mathscr{B}$, the Cauchy stress is given by

$$\boldsymbol{\sigma}(t) = \boldsymbol{P}(t)\boldsymbol{F}(t)^T / J(t), \tag{3.181}$$

where $\boldsymbol{P}(t)$ is the solution of constitutive initial value Problem 3.1 (page 76) with prescribed deformation gradient

$$\boldsymbol{F}(t) = \nabla_p \boldsymbol{\varphi}(\boldsymbol{p}, t) = \boldsymbol{I} + \nabla_p \boldsymbol{u}(\boldsymbol{p}, t). \tag{3.182}$$

The problem can be equivalently formulated in the reference configuration of $\mathscr{B}$ in terms of the material version of the principle of virtual work (3.173). For completeness, we state the material version of the fundamental initial boundary value problem in the following.

Problem 3.4 (The material quasi-static initial boundary value problem). *Find a kinematically admissible displacement function, $\boldsymbol{u} \in \mathscr{K}$, such that, for all $t \in [t_0, T]$,*

$$\int_\Omega [\boldsymbol{P}(t) : \nabla_p \boldsymbol{\eta} - \bar{\boldsymbol{b}}(t) \cdot \boldsymbol{\eta}] \, \mathrm{d}v - \int_{\partial\Omega_t} \bar{\boldsymbol{t}}(t) \cdot \boldsymbol{\eta} \, \mathrm{d}a = 0, \quad \forall\, \boldsymbol{\eta} \in \mathscr{V}, \tag{3.183}$$

where

$$\mathscr{V} = \{\boldsymbol{\eta} : \Omega \to \mathscr{U} \mid \boldsymbol{\eta} = \mathbf{0} \text{ on } \partial\Omega_u\} \tag{3.184}$$

and the Piola–Kirchhoff stress, $\boldsymbol{P}(t)$, is the solution of initial value Problem 3.1 with prescribed deformation gradient (3.182).

3.7.2. THE INFINITESIMAL PROBLEM

Under infinitesimal deformations, the quasi-static initial boundary value problem is based on the weak form (3.176). It is stated in the following.

Problem 3.5 (The infinitesimal quasi-static initial boundary value problem). *Find a kinematically admissible displacement, $\boldsymbol{u} \in \mathscr{K}$, such that, for $t \in [t_0, T]$,*

$$\int_\Omega [\boldsymbol{\sigma}(t) : \nabla \boldsymbol{\eta} - \boldsymbol{b}(t) \cdot \boldsymbol{\eta}] \, \mathrm{d}v - \int_{\partial\Omega_t} \boldsymbol{t}(t) \cdot \boldsymbol{\eta} \, \mathrm{d}a = 0, \quad \forall\, \boldsymbol{\eta} \in \mathscr{V}, \tag{3.185}$$

where

$$\mathscr{V} = \{\boldsymbol{\eta} : \Omega \to \mathscr{U} \mid \boldsymbol{\eta} = \mathbf{0} \text{ on } \partial\Omega_u\} \tag{3.186}$$

and, at each point $\boldsymbol{p}$, $\boldsymbol{\sigma}(t)$ is the solution of the constitutive initial value Problem 3.2 (page 76) with prescribed strain

$$\boldsymbol{\varepsilon}(t) = \nabla^s \boldsymbol{u}(\boldsymbol{p}, t). \tag{3.187}$$